Further results on controllable graphs

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Abstract

Connected graphs whose eigenvalues are mutually distinct and main are called controllable graphs. In recent work their relevance in control theory is recognized, and a number of theoretical and computational results are obtained. In this paper, some criteria for non-controllability of graphs are considered, and certain constructions of controllable graphs are given. Controllable graphs whose index does not exceed a given constant (close to 2.0366) are limited as part of two specific families of trees, and controllable graphs with extremal diameter are discussed. Some computational results are presented, along with corresponding theoretical observations.

Keywords: main eigenvalues, controllable graphs, constructions of graphs, small index

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1. Introduction

All graphs considered are simple and undirected. For a graph G on n vertices its characteristic polynomial P_G is just the characteristic polynomial of its adjacency matrix A $(= A(G))$. The eigenvalues of G are the roots of its characteristic polynomial, and they are usually denoted by λ_1 $(= \lambda_1(G)) \geq \lambda_2$ $(= \lambda_2(G)) \geq \cdots \geq \lambda_n$ $(= \lambda_n(G))$. The largest eigenvalue is usually called the graph index (or spectral radius).

For two graphs G_1 and G_2 we define $G_1 \cup G_2$ to be their disjoint union, while kG denotes disjoint union of k copies of G. The join $G_1 \nabla G_2$ is a graph obtained by joining every vertex of G_1 with every vertex of G_2. The cone

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over G is a join of G and the trivial graph K_1. If a graph has a trivial automorphism group then it is said to be asymmetric. Otherwise, it is symmetric. For other basic notions, the reader is referred to [4].

The tree T_n^k is obtained by taking an $(n-1)$-vertex path P_{n-1} with vertices enumerated by $1, 2, \ldots, n-1$ (in natural order) and by attaching a pendant edge at vertex enumerated by k ($1 < k < n-1$). Similarly, the tree $T_n^{k,l}$ is obtained by attaching a pendant edge at vertices enumerated by k and l ($1 < k < l < n-2$) of P_{n-2}. Both trees belong to a class of caterpillars – the trees in which a removal of all pendant vertices makes it a path.

An eigenvalue of a graph is called main if the corresponding eigenspace contains a vector for which the sum of coordinates is different from 0, while the connected graphs in which all eigenvalues are mutually distinct and main are called *controllable*. Here and in some previous work [5, 6], connectedness is a prerequisite for a controllable graph, but in general this condition can be avoided [10].

Any controllable graph is asymmetric, K_1 is controllable, while there are no other connected controllable graphs on fewer than 6 vertices [5]. Further theoretical and computational results concerning controllable graphs along with possible applications in control theory are given in [5, 6]. These applications are considered in details in [10]. The main connection between these graphs and the specific control theory problems (explaining the name: controllable graphs) will not be repeated here since this paper is concerned only by theoretical results; it can be found in any of the three papers mentioned.

It is conjectured that almost all graphs are controllable [5], and for example there are more than 5.5 millions controllable graphs with 10 vertices. On the other hand, for many asymmetric graphs it is not easy to say whether they are controllable or not without computing their spectra and eigenvectors. This motivates us to give some criteria for non-controllability and to consider the possible constructions of controllable graphs, and these are precisely the subjects of Section 2. Since controllable graphs are closely related to their spectra, such graphs with bounded least or second largest eigenvalue are considered in [6], while in Section 3 we consider those with minimal index. There we describe the structure of all controllable graphs whose index does not exceed the given constant ζ (approximately equal to 2.0366). We also consider controllable graphs with extremal diameter. Some computational results and theoretical observations are given in Appendix.

2. Constructions of controllable graphs

We first give some criteria for non-controllability of some graphs.

Theorem 1. *The following statements hold:*

(i) If a graph G has eigenvalue λ such that $-1-\lambda$ is the eigenvalue of its complement $\overline{G}$, then none of these graphs is controllable.

(ii) If a tree T contains an induced subgraph H_i equal to kP_i $(k \geq 1,\ i = 2,3,4)$ such that

(a) for $i = 2$, a vertex which does not belong to H_2 is either non-adjacent to any vertex of H_2 or it is adjacent to an even number of such vertices,

(b) for $i = 3$, a vertex which does not belong to H_3 is either non-adjacent to any vertex of H_3 or it is adjacent to a set of vertices belonging to H_3 such that an even number of them are pendant vertices of the corresponding paths,

(c) for $i = 4$, a vertex which does not belong to H_4 is either non-adjacent to any vertex of H_4 or it is adjacent only to an even number of pendant vertices of the corresponding paths,

then T is not controllable.

Proof. (i) If λ is an eigenvalue of G then $-1-\lambda$ is not a main eigenvalue of $\overline{G}$ [8], and so $\overline{G}$ is not controllable. Since G and $\overline{G}$ have the same number of main eigenvalues [5], neither is G.

(ii) (a) This structure of T gives rise to an eigenvector for non-main eigenvalue -1 defined as follows. All its entries are zero except those corresponding to vertices of H_2. There, the vertices of any P_2 correspond to entries equal to 1 and -1 such that any vertex which does not belong to H_2 is adjacent to equal number of vertices corresponding to 1, or -1. This arrangement can be realized since T is a tree, and none vertex outside H_2 is adjacent to an odd number of vertices of H_2.

(b) and (c) Using the similar reasoning we get that 0 and $\frac{\sqrt{5}-1}{2}$ are the non-main eigenvalues in the first and the second case, respectively. The eigenvector entries corresponding to the vertices of any P_3 (resp P_4) are taken to be $1, 0$ and -1 (resp. $1, \frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}$, and -1) in natural order.

The proof is complete.

We now prove the following result.

Theorem 2. *Given a controllable graph G_0 such that 0 is not an eigenvalue neither of G_0 nor $K_1\nabla\overline{G_0}$. Then each graph defined by the recurrence relation $G_{i+1} = K_1\nabla(K_1 \cup G_i)$ $(i = 0, 1, \ldots)$ is controllable.*

Proof. We prove the result by induction on i.

(a) Putting $i = 0$, since both K_1 and G_0 are controllable and G_0 has no 0 as an eigenvalue, we get that $\overline{K_1 \cup G_0}$ is controllable. Next, since 0 is not an eigenvalue of $\overline{K_1 \cup G_0} = K_1\nabla\overline{G_0}$, $\overline{K_1}$ and $\overline{K_1 \cup G_0}$ have disjoint spectra. Applying [5, Proposition 3], it follows that $G_1 = K_1\nabla(K_1 \cup G_0)$ is controllable.

(b) Let us assume, for $i \geq 1$, that G_{i+1} is obtained by the recurrence relation, where G_i is a controllable graph for which

(i) 0 is not an eigenvalue of G_i, and

(ii) 0 is not an eigenvalue of $K_1\nabla\overline{G_i}$

is fulfilled. Now we prove that (b.1) G_{i+1} is controllable graph such that (b.2) the conditions (i) and (ii) with $i + 1$ instead of i are fulfilled.

(b.1) This part is similar to (a).

(b.2) To prove that 0 is not an eigenvalue of G_{i+1} we write its adjacency matrix as

$$A(G_{i+1}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & & & & \\ \vdots & \vdots & & A(G_i) & & \\ 0 & 1 & & & & \end{pmatrix}.$$

By applying the Laplacian development along the first row of $\det(A(G_{i+1}))$ we get

$$\det(A(G_{i+1})) = -\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & A(G_i) & \\ 0 & & & \end{vmatrix} = -\det(A(G_i)) \neq 0.$$

In other words, 0 is not an eigenvalue of G_{i+1}.

Next, using the formula for the join of two graphs [4, Theorem 2.1.5] and bearing in mind that 0 is the (only) eigenvalue of K_1, we get

$$P_{K_1\nabla\overline{G_{i+1}}}(0) = P_{\overline{G_{i+1}}}(0) - (-1)^{n_{i+1}}P_{G_{i+1}}(-1).$$

where n_{i+1} is the number of vertices of G_{i+1}. Since G_{i+1} is a cone, $\overline{G_{i+1}}$ contains an isolated vertex and thus $P_{\overline{G_{i+1}}}(0) = 0$. Moreover, since G_{i+1} is controllable, 0 is a main eigenvalue of $\overline{G_{i+1}}$, and then, by Theorem 1 (i), we get that -1 cannot be (the main) eigenvalue of G_{i+1}, i.e. $P_{G_{i+1}}(-1) \neq 0$, which implies $P_{K_1\nabla\overline{G_{i+1}}}(0) \neq 0$.

The proof is complete.

The above theorem can be generalized in the following way.

Theorem 3. *Given two controllable graphs G_0 and H such that 0 is an eigenvalue of $\overline{H}$ and it is not an eigenvalue of $\overline{H \cup G_0}$. If $G_{i+1} = K_1\nabla(H \cup G_i)$ $(i = 0, 1, \ldots)$ and the eigenvalues of $H \cup G_i$ are mutually distinct and main, then G_{i+1} is controllable.*

Proof. Similarly to the previous theorem, G_{i+1} is controllable whenever 0 is not an eigenvalue of $\overline{H \cup G_i}$ and the eigenvalues of $H \cup G_i$ are mutually distinct and main. For $i = 0$ both conditions are fulfilled, while for $i \geq 1$ it remains to check the first condition. We get $P_{\overline{H\cup G_i}}(0) = P_{\overline{H}\nabla\overline{G_i}}(0) \neq 0$, where the inequality follows from the fact that 0 is an eigenvalue of both $\overline{H}$ and $\overline{G_i}$, and then -1 is not an eigenvalue of H nor G_i.

The proof is complete.

We now give some examples of controllable graphs constructed.

Example 1. Theorem 2 enables us to construct infinite families of controllable graphs starting from any such graph that satisfies two simple conditions given therein. For example, 7 out of 8 controllable graphs on 6 vertices satisfies both conditions, and one of these graphs is obtained by adding a single vertex to a 5-vertex path along with two edges joining it with two adjacent vertices of degree 2 in the path. Its spectrum is $[2.33, 1.10, 0.27, -0.59, -1.37, -1.74]$. Using this graph and the recurrence relation of Theorem 2 we get one controllable graph for any even order greater than 6. Spectra of the first 5 of these graphs are given in Table 1.

n	spectrum
8	$3.86, 1.17, 0.54, 0.16, -0.57, -1.31, -1.72, -2.11$
10	$5.28, 1.20, 0.67, 0.34, 0.15, -0.57, -1.29, -1.37, -1.73, -2.69$
12	$6.66, 1.23, 0.83, 0.47, 0.26, 0.15, -0.57, -1.25, -1.32, -1.55, -1.74, -3.18$
14	$8.00, 1.29, 1.01, 0.56, 0.35, 0.24, 0.15, -0.57, -1.23, -1.30, -1.37, -1.71, -1.77, -3.64$
16	$9.33, 1.43, 1.13, 0.65, 0.44, 0.29, 0.23, 0.15, -0.57, -1.22, -1.28, -1.32, -1.47, -1.73, -1.96, -4.09$

Table 1: Spectra of graphs $G_{i+1} = K_1\nabla(K_1 \cup G_i)$ $(i = 0, 1, \ldots, 5)$, where G_0 ia a graph on 6 vertices described in Example 1.

Example 2. It is known that for each positive integer n $(n \geq 2)$ there exists exactly one connected graph F_n of order n with property that for every pair of distinct vertices u and v, $\deg(u) \neq \deg(v)$, with exactly one exception [2]. In fact, the degree sequence of F_n is $1, 2, \ldots, \lfloor n/2 \rfloor - 1, \lfloor n/2 \rfloor, \lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \ldots, n-2, n-1$. None of these graphs is asymmetric and therefore none of them is controllable, but using them we can construct another family consisting only of controllable graphs.

For n odd and $n \geq 7$, let F'_n is obtained from F_n by joining exactly one of vertices of degree $\lfloor n/2 \rfloor$ with the vertex of degree $\lfloor n/2 \rfloor - 2$. For n even and $n \geq 8$, let F'_n is obtained from F_n by replacing edge between one vertex of degree $n/2$ and the vertex of degree $n/2+1$ by an edge joining the other vertex of degree $n/2$ with the vertex of degree $n/2-3$. Then, all graphs F'_n $(n \geq 7)$ are controllable. Namely, we have $F'_{n+2} = K_1\nabla(K_1 \cup F'_n)$, and we only have to check whether F'_7 and F'_8 satisfy the conditions of Theorem 2 which they do. In other words, an easy perturbation on graph F_n gives a controllable graph for any n $(n \geq 7)$.

Example 3. The application of Theorem 4 along with simultaneous checking whether the spectra of $H \cup G_i$ consists of mutually distinct eigenvalues can provide various controllable graphs. For example, if $G_0 = T_8^3$ and $H = \overline{T_7^3}$ (both are controllable, 0 is an eigenvalue of $\overline{H}$ and it is not an eigenvalue of $\overline{H \cup G_0}$) then for $i \geq 1$, G_i has $8(i+1)$ vertices, and by computer search we did not found that any such graph is not controllable.

In [5] the controllable NEPSes of graphs are considered, as well as the special cases: the sum $G + H$ and the product $G \times H$ (if necessary, see [4, p. 44] for definitions). Here we give an immediate consequence of [5, Proposition 5] along with its short proof.

Theorem 4. *Given two controllable graphs G and H with eigenvalues $\lambda_1(G)$, $\lambda_2(G), \ldots, \lambda_n(G)$ and $\lambda_1(H), \lambda_2(H), \ldots, \lambda_m(H)$, respectively. The sum $G+H$*

is controllable whenever it is connected and $\lambda_i(G) - \lambda_j(G) \neq \lambda_k(H) - \lambda_l(H)$ *holds for every* $1 \leq i < j \leq n$ *and* $1 \leq k < l \leq m$*. The product* $G \times H$ *is controllable if either both* G *and* H *are equal to* K_1*, or they do not contain an eigenvalue equal to 0,* $G \times H$ *is connected, and* $\lambda_i(G)/\lambda_j(G) \neq \lambda_k(H)/\lambda_l(H)$ *holds for every* $1 \leq i < j \leq n$ *and* $1 \leq k < l \leq m$*.*

PROOF. All eigenvalues of $G+H$ or $G \times H$ are main [4], and so it is sufficient to check whether they are mutually distinct. The proof follows from the fact that the eigenvalues of $G+H$ (resp. $G \times H$) are $\lambda_i(G) + \lambda_k(H)$ (resp. $\lambda_i(G) \cdot \lambda_k(H)$), $1 \leq i \leq n,\ 1 \leq k \leq m$.

We consider another graph composition. The *corona* $G \circ H$ of graphs G and H is obtained by taking G (having n vertices) and the graph nH, and then joining the i–th vertex of G to every vertex in the i–th copy of H $(i = 1, \dots, n)$.

Theorem 5. *Given a graph* G *distinct from* K_1*, then* $G \circ K_1$ *is controllable if and only if* G *is controllable and 0 is not its eigenvalue.*

PROOF. We can order the vertices of $G \circ K_1$ such that its adjacency matrix has the form

$$A(G \circ K_1) = \begin{pmatrix} A(G) & I_n \\ I_n & O_n \end{pmatrix},$$

where I_n and O_n denote the unit and the zero matrix of order n, respectively.

Denote the eigenvalues of G by $\lambda_1, \dots, \lambda_n$ and the corresponding eigenvectors by $e(\lambda_1) = (x_1^{(1)}, \dots, x_n^{(1)})^T, \dots, e(\lambda_n) = (x_1^{(n)}, \dots, x_n^{(n)})^T$. Then, by Theorem 4.2 of [9], the eigenvalues of $G \circ K_1$ are the solutions of the equations

$$\mu^2 - \lambda_i \mu - 1 = 0 \ (i = 1, \dots, n). \tag{1}$$

If μ_{ij} $(j = 1, 2)$ are the eigenvalues of $G \circ K_1$ obtained by putting λ_i $(i = 1, \dots, n)$ into (1) then it is easy to verify that $\left(x_1^{(i)}, \dots, x_n^{(i)}, \frac{x_1^{(i)}}{\mu_{ij}}, \dots, \frac{x_n^{(i)}}{\mu_{ij}}\right)^T$ $(i = 1, \dots, n;\ j = 1, 2)$ are the corresponding eigenvectors (note that, by (1), 0 is not an eigenvalue of $G \circ K_1$, so the eigenvectors are well defined).

Now, by (1), the eigenvalues of G are mutually distinct if and only if the eigenvalues of $G \circ K_1$ are mutually distinct. Computing the sums of entries of the corresponding eigenvectors of $G \circ K_1$ we get

$$\left(1+\frac{1}{\mu_{ij}}\right)\left(x_1^{(i)}+\ldots+x_n^{(i)}\right),\quad i=1,\ldots,n;\ j=1,2,$$

and therefore, the eigenvalues of $G\circ K_1$ are main if and only if the same hold for the eigenvalues of G and -1 is not an eigenvalue of $G\circ K_1$ or, by (1), 0 is not an eigenvalue of G.

The proof is complete.

We have the following corollary.

Corollary 1. *If G is a controllable graph on n vertices then the number of its pendant vertices does not exceed $\frac{n}{2}$. In addition, if this bound is attained, then $G=G'\circ K_1$, where G' is a controllable graph and 0 is not its eigenvalue.*

Proof. If the number of pendant vertices is greater than $n/2$ then there is at least one pair of such vertices with the common single neighbour (it is usually said that such vertices are duplicate). Then, by [6, Lemma 1.1 (i)], G is not controllable.

If G has exactly $n/2$ pendant vertices (evidently n must be even), using the above argumentation based on duplicate vertices we get that G must be a corona (i.e. $G=G'\circ K_1$). The reminder of the proof follows from the previous theorem.

3. Controllable graphs with small index

Denote $\zeta=\sqrt{\frac{14+\sqrt[3]{188+12\sqrt{93}}+\sqrt[3]{188-12\sqrt{93}}}{6}}$ (≈ 2.0366). We consider controllable graphs whose index does not exceed this constant.

It is known that any graph whose index does not exceed 2 is a subgraph of some Smith graph [4, Theorem 3.11.1]. Considering the Smith graphs and their connected subgraphs we find that almost all of them are symmetric. The exceptions are K_1 and T_n^3 ($n=7,8,9$). Inspecting the last 3 graphs we get that the first two of them are controllable, while T_9^3 is not.

We next consider the graphs whose index is greater than 2.

Lemma 1. $\lambda_1(T_n^4)<\zeta$.

PROOF. By [4, Theorem 2.2.2], $\lambda_1(T_n^4)$ is less than the largest real root of the equation

$$\frac{1}{2}(x+\sqrt{x^2-4})(x^5-4x^3+3x)-x^2(x^2-2)=0,$$

which after some usual transformations becomes

$$\frac{1}{2}x\left(x(x^4-6x^2+7)+\sqrt{x^2-4}(x^4-4x^2+3)\right)=0.$$

The largest real root of the above equation is equal to the largest real root of

$$\sqrt{x^2-4}(x^4-4x^2+3)=-x(x^4-6x^2+7).$$

Squaring and simplifying this equation we get

$$7x^4+9=x^2(x^4+14).$$

Putting $x^2=t$, we get the cubic equation with exactly one real root which correspond to the largest real root of the initial equation. The cubic equation is easily solved, its real root is equal to ζ^2, and the proof is complete.

Lemma 2. *Let G be a connected graph with $2 \le \lambda_1(G) \le \zeta$, then G is one of the following graphs*

1. T_n^3 $(n \ge 10)$ *or* T_n^4 $(n \ge 9)$,
2. T_{10}^5 *or* T_{11}^5,
3. T' *– a tree obtained by identifying a middle vertex of* P_5 *with a pendant vertex of* P_4,
4. $T_n^{2,n-4}$ $(n \ge 7)$ *or* $T_n^{3,n-4}$ $(n \ge 15)$.

PROOF. Let G have n vertices, then it is a tree of diameter $n-3$ or $n-2$ [3].

Assume first that diameter of G is equal to $n-2$. Then G is equal to T_n^k, where $k \ge 3$ (since otherwise G is a subgraph of the corresponding Smith graph causing $\lambda_1(G) < 2$). For $k=4$, by the previous lemma, we have that the index of any graph T_n^4 is less than ζ. In addition, for $n \ge 9$ the index is greater than 2. Using the eigenvalue interlacing (see [4, Corollary 1.3.12]), we get that $\lambda_1(T_n^3) < \zeta$, while for $n \ge 10$ we get $\lambda_1(T_n^3) > 2$. Putting $k \ge 5$, and with no loss of generality $n \geq 2k$, we get only two solutions: T_{10}^5 and T_{11}^5.

Assume now that diameter of G is equal to $n-3$. If G is not a caterpillar then it is obtained by identifying a pendant vertex of P_3 with an interior vertex of some path. Inspecting such trees we get that only T' satisfies our spectral condition.

If G is a caterpillar then it cannot contain a vertex of degree 4 (since then $\lambda_1(G) \notin (2, \zeta)$), and thus it is equal to $T_n^{k,l}$. Since $\lambda_1(G) > 2$, we can suppose that $2 \leq k < l < n-4$. Similarly to the previous lemma we get that $\lambda_1(T_n^{2,n-5}) \to \zeta$, when $n \to \infty$, but here the index is greater than ζ for any $n \geq 7$. Due to eigenvalue interlacing, it remains to consider the graphs $T_n^{2,n-4}$ and $T_n^{3,n-4}$. Using the result concerning the index of graphs with an internal path (cf. [7]) we get $\lambda_1(T_n^{2,n-4}) \in (2, \zeta)$ for $n \geq 7$ and $\lambda_1(T_n^{3,n-4}) \in (2, \zeta)$ for $n \geq 15$.

Collecting the graphs obtained we get the proof.

In [3] the graphs with index in the interval $(2, \sqrt{2+\sqrt{5}})$ are considered. In the previous lemma we used the opportunity to determine all connected graphs with index in the subinterval $(2, \zeta)$, and now consider the controllability of graphs whose index does not exceed ζ.

Theorem 6. *Let G be an n-vertex controllable graph whose index does not exceed ζ, then G is K_1 or it is equal to T_n^3 or T_n^4.*

Proof. Controllable graphs whose index is at most 2 are discussed at the origin of the section. Considering the graph of Lemma 2, by direct computation, we get that T_{11}^5 and T' are not controllable, while the graphs $T_{10}^5, T_n^{2,n-4}$ $(n \geq 7)$ and $T_n^{3,n-4}$ $(n \geq 15)$ are symmetric, and therefore they are not controllable neither, and the proof is complete.

We now give more details about the controllability of T_n^k $(k = 3, 4)$. Let λ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be its eigenvalue and the corresponding eigenvector. The entries of $\mathbf{x}$ satisfy the following system:

$$\begin{aligned} x_2 &= \lambda x_1 \\ x_1 + x_3 &= \lambda x_2 \\ &\cdots \end{aligned}$$

$$
\begin{aligned}
x_{k-2} + x_k &= \lambda x_{k-1} \\
x_{k-1} + x_{k+1} + x_n &= \lambda x_k \\
x_k + x_{k+2} &= \lambda x_{k+1} \\
&\cdots \\
x_{n-3} + x_{n-1} &= \lambda x_{n-2} \\
x_{n-2} &= \lambda x_{n-1} \\
x_k &= \lambda x_n
\end{aligned} \tag{2}
$$

Theorem 7. *T_n^3 is not controllable whenever $n \leq 7$, or $n \in \{5+4l, 6+3l, 9+5l : l \in \mathbb{N}\}$.*

Proof. If $\lambda = 0$ then n must be odd, and solving the system (2) we get $x_j = 0$ whenever j is odd and $j \neq n$; $x_2 = 0$, $x_4 = -x_n$ and $x_{2j+2} = -x_{2j+4}, j = 1, 2, \ldots, \frac{n-5}{2}$. Summing the entries obtained we get $\sum_{j=1}^{n} x_j = 0$ whenever $n = 5 + 4l,\ l \in \mathbb{N}$, i.e. 0 is a non–main eigenvalue of T_n^3, and consequently this tree is not controllable, whenever $n = 5 + 4l,\ l \in \mathbb{N}$.

If $\lambda \neq 0$ considering the system (2) we get

$$
x_2 = \lambda x_1,\ x_3 = (\lambda^2 - 1)x_1,\ x_n = \frac{\lambda^2 - 1}{\lambda} x_1,\ x_4 = \frac{\lambda^4 - 3\lambda^2 + 1}{\lambda^2} x_1. \tag{3}
$$

We now distinguish two cases depending on x_3:

Case 1: $x_3 = 0$. This implies $x_1 = 0$ (but then all the remaining x_j's are also equal to zero) or $\lambda = \pm 1$. Solving the system we get that -1 and 1 are eigenvalues of T_n^3 if and only if $n = 6 + 3k$; in addition, -1 is always non–main.

Case 2: $x_3 \neq 0$. Assume that $x_4 = 0$ then

$$
\lambda^4 - 3\lambda^2 + 1
$$

must be equal to zero (note that this is just the characteristic polynomial of P_4). We get $\lambda \in \left\{\pm\frac{\sqrt{5}-1}{2}, \pm\frac{\sqrt{5}+1}{2}\right\}$. These numbers are contained in the spectrum of T_n^3 if and only if $n = 9 + 5l, l \in \mathbb{N}$; in addition, $\frac{\sqrt{5}-1}{2}$ is always non–main.

Summarizing the facts obtained we get the result and the proof is complete.

In the similar way we prove the following theorem.

Theorem 8. *T_n^4 is not controllable whenever $n \leq 7$, n is even, or $n = 11 + 6l, l \in \mathbb{N}$.*

PROOF. First, T_n^4 is not controllable whenever n is even since zero is its eigenvalue of the multiplicity 2. Next, using the reasoning similar to the previous theorem we get that -1 and 1 are non-main eigenvalues if $n = 11 + 6l,\ l \in \mathbb{N}$, and the proof follows.

Except from those graphs given in the previous two theorems, using the computer, we did not found any non-controllable trees of the form T_n^3 or T_n^4. A detailed discussion concerning the controllability of T_n^3 is given in the remark below.

Remark 1. Since all eigenvalues of T_n^3 are mutually distinct, in order to determine whether this tree is controllable or not one should consider the remaining case in the proof of the previous theorem: $x_3, x_4 \neq 0$. Considering the equations of (2) we get

$$(\lambda - 2)\sum_{j=1}^{n} x_j = x_3 - x_1 - x_{n-1} - x_n.$$

Using the expressions of x_3 and x_n (given in (3)) we get

$$(\lambda - 2)\sum_{j=1}^{n} x_j = \frac{\lambda^3 - \lambda^2 - 2\lambda + 1}{\lambda} x_1 - x_{n-1}.$$

Since 2 is not an eigenvalue of T_n^3 except for $n = 9$ (which can easily be confirmed by the eigenvalue interlacing), choosing $x_1 = 1$ we get that the eigenvalue λ is non-main if $x_{n-1} = \frac{\lambda^3-\lambda^2-2\lambda+1}{\lambda} = f(\lambda)$.

Considering the system of the recurrence equations $x_{j-1}+x_{j+1} = \lambda x_j,\ j = 4, \ldots, n-2$ (this system is derived from (2)) along with the initial conditions (the second and the last equation of (3)) we can obtain the exact value of x_{n-1} (in terms of λ and n); denote it by $x_{n-1} = g(\lambda, n)$ (we will not write its full expression since it is complicate and will not be used). So, besides the cases given in Theorem 7, T_n^3 is not controllable if at least one its eigenvalue λ satisfies $f(\lambda) = g(\lambda, n)$. We found that these functions coincide in some points for any sufficiently large n, but for $n \leq 1000$ we did not found a situation in which any of these points is an eigenvalue of the corresponding tree.

Using the computer search we found that for any n ($7 \leq n \leq 1000$) there is at least one controllable graph of order n and diameter $n-2$. We get that T_n^k ($7 \leq n \leq 1000$) is controllable for at least one value k ($3 \leq k \leq 10$), unless $n = 60l - 1, n = 210l$, or $n = 461$. In these exceptional cases the controllability is obtained for at least one k greater than 10.

Since except from K_1 any path is not controllable, in this way we get the controllable graphs with the largest possible diameter for any n ($7 \leq n \leq 1000$). According to this we state the following conjecture.

Conjecture 1. *For any $n \geq 7$, there is a controllable graph of order n and diameter $n-2$.*

Regarding the least diameter, since there are no complete controllable graphs (distinct from K_1) it cannot be equal to 1, but it is easy to determine many controllable graphs whose diameter is equal to 2. Namely, the cone over G is controllable whenever G is controllable graph whose complement avoids 0 in the spectrum [5]; in addition its diameter is equal to 2. In this context we can establish the following result.

Theorem 9. *Let T be a controllable tree distinct form K_1, then $\overline{T}$ is controllable graph whose diameter is equal to 2.*

Proof. First, $\overline{T}$ is controllable since the connected complement of any controllable graph is controllable. To prove that its diameter is equal to 2 it is sufficient to show that any pair of adjacent vertices in T have at least one common non–adjacent vertex. This follows from the fact that any controllable graph is asymmetric and that there are no controllable trees (distinct from K_1) with fewer than 7 vertices.

Appendix

In Table 2 we give some computational data on controllable graphs which belong to some specific classes. Recall that a connected graph is called highly irregular if every its vertex is adjacent only to vertices with mutually distinct degrees. These graphs are studied in [1]. Many of them are asymmetric and therefore they are candidates for controllable graphs.

n	$T(n)$	$I(n)$	$C(n)$	$P(n)$	$CP(n)$	$E(n)$	$CE(n)$	$HI(n)$	$CHI(n)$
6	112	8	8	99	8	8	0	1	0
7	853	144	85	646	80	37	0	0	0
8	11117	3552	2275	5974	1530	184	0	3	0
9	261080	131452	83034	71885	24838	1782	184	3	0
10	11716571	7840396	5512583	1052805	448991	31026	6293	13	2
11	1006700565	797524380	?	17449299	?	1148626	458891	21	7

Table 2: The numbers of simple connected graphs $T(n)$, asymmetric graphs $I(n)$, controllable graphs $C(n)$, connected planar graphs $P(n)$, planar controllable graphs $CP(n)$, connected Eulerian graphs $E(n)$, Eulerian controllable graphs $CE(n)$, highly irregular graphs $HI(n)$, and highly irregular controllable graphs $CHI(n)$, on n $(6 \leq n \leq 11)$ vertices.

We also compute the data on highly irregular controllable graphs on at most 15 vertices. Here are the total numbers of connected highly irregular graphs on 12, 13, 14, and 15 vertices, respectively (the numbers of controllable graphs are given in brackets): 110 (17), 474 (247), 2545 (962), and 18696 (10209).

We conclude by the following simple facts concerning highly irregular controllable graphs.

- The highest possible degree in highly irregular controllable graph is $\lfloor n/2 \rfloor - 1$ (the highest possible degree in any highly irregular graph is $\lfloor n/2 \rfloor$ [1], but if so then the graph is not asymmetric; highly irregular controllable graphs with vertices of degree $\lfloor n/2 \rfloor - 1$ can easily be found).

- Highly irregular controllable graph with minimum number of vertices (distinct from K_1) has 10 vertices; there are 2 such graphs (computational result).

- There are no highly irregular controllable graphs with maximum vertex degree 2 (simple observation). Highly irregular controllable tree with minimum number of vertices (distinct from K_1) has 13 vertices; this is $T_{13}^{3,4}$ (confirmed by computer).

- Highly irregular controllable graph with exactly k vertices with maximum degree d has at least $\frac{1}{2}d(d+1)k$ vertices (any highly irregular graph with less number of vertices is symmetric).

- Highly irregular controllable tree with exactly k vertices with maximum degree d has at least $2^{d-1+k/2}$ vertices (same as above).

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