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A note on the minimum rank of graphs with given dominating induced subgraph

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Abstract

An induced subgraph of a graph G is said to be dominating if every vertex of G is at distance at most one from this subgraph. We investigate pairs (G, F) where F is a nonsingular dominating induced subgraph of G, and the rank of G (that is, the rank of its adjacency matrix) attains the minimum, i.e., equals the number of vertices in F. It turns out that the inverse of the adjacency matrix of a nonsingular path, half graph, or even cycle is the adjacency matrix of a related signed graph; here, a half graph refers to a connected chain graph with exactly one vertex in each cell. We exploit this property to give a complete characterization of graphs G paired with any of these graphs in the role of F. The bipartite case is singled out. It occurs that every nonsingular F is paired with an infinite family of graphs G, and their number is comparatively large even if we exclude the existence of the so-called twin vertices. The latter empirical observation is demonstrated through some examples.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. Simultaneously, we deal with signed graphs, a generalization in which every edge is declared positive or negative. The number of vertices of a graph $G = (V, E)$ is called the *order*. By the *eigenvalues*, the *spectrum* and the *rank* $\operatorname{rank}(G)$ of G, we mean the eigenvalues, the spectrum and the rank of its $\{0, 1\}$-adjacency matrix A_G. The adjacency matrix of a signed graph is obtained by reversing the sign of every entry that corresponds to a negative edge.

A graph G is *singular* if its adjacency matrix is singular. An induced subgraph, say H, of G is called *dominating* if every vertex of G is in H or has a neighbor in H. In other words, H is dominating if and only if every vertex of G is at distance at most 1 from H. Non-adjacent vertices that share the same set of neighbors are called *twins*.

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We know from [6, Theorem 5.1.6] that every graph G of rank k contains a nonsingular induced subgraph of order k. Moreover, if G is connected, then there is a connected dominating induced subgraph with the same property. This establishes a method to study the rank of graphs by considering fixed nonsingular dominating induced subgraphs and their interplay with the remaining vertices. In particular, characterization of graphs with fixed rank can be considered in this way.

The path and the cycle of order n are denoted by P_n and C_n, respectively. A *half graph* H_{2n} is a bipartite graph with color classes $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$, such that a vertex u_i is adjacent to vertices $v_1, v_2, \dots, v_{n+i-1}$, for $1 \leq i \leq n$. The reader may recognize that half graphs are a particular subclass of the so-called chain graphs.

One may verify that every set of k ($k \geq 2$) twin vertices gives rise to the eigenvalue 0 with multiplicity $k-1$. This means that every nonsingular graph of order n is a dominating induced subgraph in an infinite family of graphs with rank n; it is sufficient to add twin vertices, one-by-one. If we exclude twin vertices, then the number of resulting graphs is finite (see [6, Proposition 5.1.4]), but often comparatively large as we demonstrate in few examples.

We mention that all graphs with rank at most 9 are characterized in series of publications [7, 8, 14, 15, 19]. Another characterization of graphs with rank at most 5 is given in [3, 4]. In contrast to these references, we do not fix the rank, but fix the structure of a maximal nonsingular dominating subgraph. Precisely, we consider P_{2n}, C_{4n+2} or H_{2n} in the role of such a subgraph, where the specified orders guarantee the nonsingularity. The motivation lies in the fact the inverse of the corresponding adjacency matrix is the adjacency matrix of a fixed related signed graph. We exploit this result to offer a characterization of graphs that contain any of these graphs as a dominating induced subgraph and have the minimum rank. The bipartite case is resolved, as well.

Considering more related results, we point out that a maximal order of connected twin-free graphs with given rank has been studied in [9, 10, 13]. Many other results relate the rank to particular graph invariants or compute the rank of graphs belonging to specified structural classes; some of them can be found in [5, 11, 12, 16] and references therein.

Additional terminology and notation are given in the forthcoming text. For notation or terminology not explained here, we refer the reader to any of [6, 17]. Our results are reported in Section 2. Some constructions are presented in Section 3.

2 Results

We quote the following result, known as the Reconstruction Theorem.

Theorem 2.1 ([6, Theorem 5.1.7]). *Let X be a set of k vertices in a graph G and suppose that G has adjacency matrix*

$$A_G = \begin{pmatrix} A_X & B^\intercal \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ

in G if and only if μ is not an eigenvalue of $G-X$ and

$$\mu I - A_X = B^\intercal(\mu I - C)^{-1}B. \tag{2.1}$$

With the notation of Theorem 2.1, let $F \cong G-X$ (where $\cong$ designates isomorphic graphs). It is clear that F is a subgraph of G induced by $\overline{X} = V(G)\backslash X$, with $|\overline{X}| = n-k$ and $A_F = C$. For $u \in X$, denote by $\mathbf{b}_u$ the vector-column of B corresponding to u. Evidently, $\mathbf{b}_u$ is the characteristic vector of the F-neighborhood $N_F(u)$ of u. In light of (2.1), we define the bilinear form on $\mathbb{R}^{n-k}$ by $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^\intercal(\mu I - C)^{-1}\mathbf{y}$. Equating the entries in the same identity, we arrive at

$$\langle \mathbf{b}_u, \mathbf{b}_v\rangle = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Theorem 2.1 tells us that if F is an induced subgraph of a graph G with the invertible adjacency matrix C, then the rank of G is $|V(F)|$ if and only if the adjacency matrix A_G is as in this theorem. In other words, the columns of B satisfy (2.2). To say more in this regard, we need more information about F, and in what follows we consider the three possibilities: when F is an even path, an even chain graph or a cycle with $4n+2$ vertices. Each of them is nonsingular.

We start with a path in the role of a dominating induced subgraph.

Theorem 2.2. *Assume that the path P_{2n} is a dominating induced subgraph in a graph G, and let its vertices be labeled by $1, 2, \dots, 2n$ in the natural order. Then* $\operatorname{rank}(G) \geq 2n$ *with equality if and only if*

(i) for every vertex $u \in V(G) \setminus V(P_{2n})$, the P_{2n}-neighborhood $N_P(u)$ consists of vertices satisfying

$$|\{i, j \ : \ i \text{ odd}, i < j, j - i \equiv 1 \ (\mathrm{mod}\, 4)\}| = |\{i, j \ : \ i \text{ odd}, i < j, j - i \equiv 3 \ (\mathrm{mod}\, 4)\}|$$

and

(ii) for every pair $u, v \in V(G) \setminus V(P_{2n})$ and every pair i, j such that either $i \in N_P(u), j \in N_P(v)$ or $i \in N_P(v), j \in N_P(u)$, it holds

$$|\{i, j \ : \ i \text{ odd}, i < j, j - i \equiv 1 \ (\mathrm{mod}\, 4)\}| - |\{i, j \ : \ i \text{ odd}, i < j, j - i \equiv 3 \ (\mathrm{mod}\, 4)\}| \in \{0, 1\},$$

with $u \nsim v$ if and only if the previous difference is 0.

Proof. Since $\operatorname{rank}(P_{2n}) = 2n$, we have $\operatorname{rank}(G) \geq 2n$. We proceed with the equality case. By Theorem 2.1, $\operatorname{rank}(G) = 2n$ if and only if the equality (2.2) holds for $u, v \in V(G) \setminus V(P_{2n})$, with $\mu = 0$ and $C = A_{P_{2n}}$.

We proceed to determine the inverse of $A_{P_{2n}}$. If the vertices of P_{2n} are labelled in the natural order, then the (i, j)-entry of its adjacency matrix is 1 if and only if $|i - j| = 1$. We claim that the (i, j)-entry of the inverse is nonzero if and only if both $i - j$ and $\min\{i, j\}$ are $\equiv 1 \ (\mathrm{mod}\, 2)$, along with

$$(i, j) = \begin{cases} 1 & \text{if } |i - j| \equiv 1 \ (\mathrm{mod}\, 4), \\ -1 & \text{otherwise.} \end{cases}$$

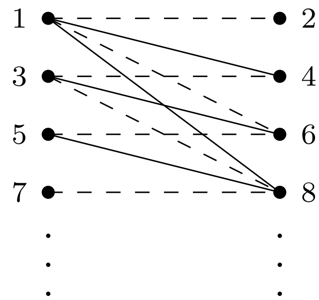

Figure 1: The signed half graph for the proof of Theorem 2.2. Negative edges are dashed.

This is not complicated to verify, as a row of $A_{P_{2n}}$ and a column of the second matrix match in at most two nonzero places, along with the desired conclusion. Accordingly, the inverse of $-A_{P_{2n}}$ appears to be the adjacency matrix of a signed half graph, say Σ, illustrated in Figure 1, where the vertex labeling is inherited from P_{2n}.

Now, $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mathbf{b}_u^\intercal A_\Sigma \mathbf{b}_u = 0$ holds if and only if the number of positive edges in the subgraph of Σ induced by $N_P(u)$ is equal to the number of negative edges in the same subgraph. We first observe that there is an edge between the vertices i and j if and only if the smaller of them is odd and the other is even (see the figure). The next observation is that the corresponding edge is positive if and only if $j - i \equiv 3 \pmod 4$, which gives (i).

Next, $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0$ holds if and only if the number of positive edges located between $N_P(u)$ and $N_P(v)$ in Σ is equal to the number of negative edges located between the same vertex sets, i.e., if and only if the difference of item (ii) is zero. Similarly, $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -1$ if and only if the same difference is 1 (i.e., the number of positive edges is the number of negative edges minus 1). In addition, in the former (resp. latter) case u and v are non-adjacent (adjacent). This completes (ii) and the entire proof. $\square$

In the previous proof, we gave an explicit construction of $A_{P_{2n}}^{-1}$. In this context, it is worth mentioning that the inverse of the adjacency matrix of a nonsingular tree is obtained in [2, Theorem 3.33], which is an alternative way to arrive at the same result. Our approach emphasizes the related signed graph which is significant for development of the paper. We proceed with the following consequence.

Corollary 2.3. *Assume that the path P_{2n+1} is a dominating induced subgraph in a graph G, and let its vertices be labeled by $1, 2, \ldots, 2n+1$ in the natural order. Then $\operatorname{rank}(G) \geq 2n$ with equality if and only if the subpath P_{2n} obtained by deleting the vertex $2n+1$ satisfies (i) and (ii) of Theorem 2.2 and for every vertex $u \in V(G) \setminus V(P_{2n+1})$ its P_{2n}-neighborhood consists of vertices satisfying*

$$|\{i \;:\; 2n - i \equiv 1 \pmod 4\}| - |\{i \;:\; 2n - i \equiv 3 \pmod 4\}| \in \{0, 1\},$$

with $u \nsim 2n+1$ if and only if the previous difference is 0.

Proof. Since $\operatorname{rank}(P_{2n+1}) = 2n$, we infer that $\operatorname{rank}(G) \geq 2n$. The equality case is considered as in the proof of Theorem 2.2 with P_{2n} in the role of a dominating path, along with an additional assumption that G contains a vertex adjacent to an endvertex of this path, in the

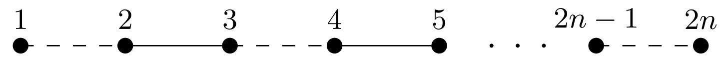

Figure 2: The signed path for the proof of Theorem 2.4.

formulation of this corollary labeled by $2n+1$. Indeed, P_{2n} also dominates G, otherwise G would contain P_{2n+2} (with rank $2n+2$) as an induced subgraph which implies $\operatorname{rank}(G) > 2n$. The remaining assumption of the corollary arises from $\langle \mathbf{b}_u, \mathbf{b}_{2n+1}\rangle \in \{0,-1\}$. □

The next instance is a half graph.

Theorem 2.4. *Assume that the half graph H_{2n} is a dominating induced subgraph in a graph G, and let its vertices be labelled as in Figure 1. Then* $\operatorname{rank}(G) \geq 2n$ *with equality if and only if*

(i) for every vertex $u \in V(G) \setminus V(H_{2n})$, the H_{2n}-neighborhood $N_H(u)$ consists of vertices satisfying

$$|\{i,j\ :\ i-j=1, i \text{ odd}\}| = |\{i,j\ :\ i-j=1, i \text{ even}\}|$$

and

(ii) for every pair $u,v \in V(G)\setminus V(H_{2n})$ and every pair i,j such that either $i \in N_H(u), j \in N_H(v)$ or $i \in N_H(v), j \in N_H(u)$, it holds

$$|\{i,j\ :\ i-j=1, i \text{ odd}\}| - |\{i,j\ :\ i-j=1, i \text{ even}\}| \in \{0,1\},$$

with $u \nsim v$ if and only if the previous difference is 0.

Proof. The adjacency matrix $A_{H_{2n}}$ is nonsingular, see [1]. Accordingly, $\operatorname{rank}(G) \geq 2n$.

For the equality case, we exploit the idea of the proof of Theorem 2.2. The matrix $-A_{H_{2n}}^{-1}$ is the adjacency matrix of the signed path illustrated in Figure 2; to see this one may follow the corresponding part of the proof of Theorem 2.2. Accordingly, $\langle \mathbf{b}_u, \mathbf{b}_u\rangle = 0$ holds if and only if the number of positive edges and the number of negative edges in the signed graph induced by $N_H(u)$ are equal, i.e., if and only if item (i) holds.

Comparing the numbers of positive and negative edges between $N_H(u)$ and $N_H(v)$, we arrive at (ii), and the proof is complete. □

And the next station is a cycle.

Theorem 2.5. *Assume that the cycle C_{4n+2} is a dominating induced subgraph in a graph G, and let its vertices be labeled by $1,2,\dots,4n+2$ in the natural order. Then* $\operatorname{rank}(G) \geq 4n+2$ *with equality if and only if*

(i) for every vertex $u \in V(G)\setminus V(H_{2n})$, the C_{4n+2}-neighborhood $N_C(u)$ consists of vertices satisfying

$$|\{i,j\ :\ |i-j| \equiv 1 \pmod 4\}| = |\{i,j\ :\ |i-j| \equiv 3 \pmod 4\}|$$

and

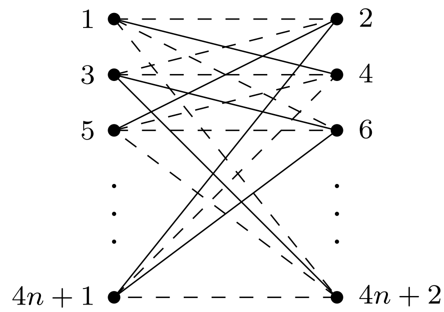

Figure 3: The signed complete bipartite for the proof of Theorem 2.5.

(ii) for every pair $u, v \in V(G) \setminus V(C_{4n+2})$ *and every pair* i, j *such that either* $i \in N_C(u), j \in N_C(v)$ *or* $i \in N_C(v), j \in N_C(u)$*, it holds*

$$|\{i, j \ : \ |i-j| \equiv 1 \pmod 4\}| - |\{i, j \ : \ |i-j| \equiv 3 \pmod 4\}| \in \{0, 2\},$$

with $u \nsim v$ *if and only if the previous difference is* 0*.*

Proof. The lower bound for the rank is evident, and we consider the equality case. By direct multiplication one may confirm that $-A^{-1}_{C_{4n+2}}$ is given by

$$(i, j) = \begin{cases} -1/2 & \text{if } |i-j| \equiv 1 \pmod 4, \\ 1/2 & \text{if } |i-j| \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, this matrix is equal to $\frac{1}{2}A_\Sigma$, where Σ is the signed complete bipartite graph of Figure 3. The remainder of the proof is a slight modification of the proof of Theorem 2.2. □

Cycles with $4n$ vertices are singular, and 0 appears in their spectra with multiplicity 2. However, each contain an induced path P_{4n-2}, and thus graphs G containing C_{4n} as a dominating induced subgraph and having rank $4n-2$ are obtained by following Corollary 2.3.

We consider the bipartite case.

Theorem 2.6. *Let* G *be a graph containing a dominating induced subgraph* F*, where* F *is a graph of either Theorem 2.2, or 2.4, or 2.5. Then* G *is bipartite with* $\operatorname{rank}(G) = |V(F)|$ *if and only if*

(a) the subgraph induced by $V(G) \setminus V(F)$ *is edgeless,*

(b) for every $u \in V(G) \setminus V(F)$ *its* F*-neighborhood* $N_F(u)$ *belong to exactly one color class of* F *and*

(c) for every $v \in V(G) \setminus V(F)$ *such that* F*-neighbors of* u *and* v *are in distinct color classes, the set* $N_F(u) \cup N_F(v)$ *consist of vertices satisfying the equality of item (i) of the corresponding theorem.*

Proof. Assume first that G is bipartite with $\mathrm{rank}(G) = |V(F)|$.

Since F is bipartite, every $u \in V(G) \setminus V(F)$ is adjacent to vertices belonging to exactly one color class of F, which gives (b).

Let v be as in item (c) of this statement. Then the difference of item (ii) of the theorem treating a specified graph F (i.e., one of Theorems 2.2, 2.4 or 2.5) is zero. Moreover, this difference reduces to the equality of item (i) of the same theorem since $N_F(u) \cap N_F(v) = \emptyset$ and $|i - j|$ is even for every pair i, j belonging to one of $N_F(u)$ or $N_F(v)$. This proves (c).

Suppose that there is an edge between $u, v \in V(G) \setminus V(F)$. This means that u and v are in distinct color classes of G, and then $|i - j|$ is even for every pair $i \in N_F(u), j \in N_F(v)$, meaning that the difference in (ii) of the corresponding theorem is zero (in fact, both minuend and subtrahend are zero), which contradicts the existence of an edge between u and v. Hence, (a) holds.

Assume now that all three items of the statement of this theorem hold. First, G is bipartite since F is bipartite, the subgraph induced by $V(G) \setminus V(F)$ is edgeless (by (a)) and its vertices are dispersed into color classes according to (b).

It remains to consider the rank of G. Item (i) of the corresponding theorem follows from (b) of this theorem. Item (ii) follows from (a) and (c). □

3 Examples

In this section we give some examples that illustrate the previous results.

Example 3.1. Let O (resp. E) denote a subset containing odd (even) vertices of the path P_{2n} such that $u > v$ holds for every pair $u \in O, v \in E$. If every $u \in V(G) \setminus V(P_{2n})$ is adjacent only to vertices of $O \cup E$ and the graph induced by $V(G) \setminus V(P_{2n})$ is edgeless, then $\mathrm{rank}(G) = 2n$.

Indeed, items (i) and (ii) of Theorem 2.2 hold trivially (as they reduce to $0 = 0$ and $0 - 0$, respectively).

Bipartite examples are easily constructed on the basis of Theorem 2.6. For instance, by choosing G without vertices of item (c) will do. In other cases, the same item establishes a crucial condition.

We rather proceed with the following concept. If F is a nonsingular graph, then there is an infinite family of graphs containing F as a dominating induced subgraph and having rank $|V(F)|$. To see this, it is sufficient to observe that in this case X of Theorem 2.1 may contain an arbitrary number of twins. However, if we restrict ourselves to twin-free graphs, then for every F there is a finite number of them (as noted in the opening section), and the results of the previous section give their structure for particular choices of F. We say that a twin-free graph G with the previous properties is *maximal* if a vertex can be added to X only if this vertex is a twin to some existing vertex. Such graph G can be seen as a maximal twin-free extension of F. Clearly, maximal twin-free extensions are of particular interest as every other twin-free extension is an induced subgraph of a maximal one. In other words, once we have obtained maximal twin-free extensions, we have all twin-free extensions.

Figure 4: Maximal twin-free extensions of P_4.

Example 3.2. The Star Complement Library (SCL) software was developed to support constructions of graphs of Theorem 2.1 when F and μ are given [18]. In particular, for $\mu = 0$ it results in maximal twin-free extensions, so it is exactly what we need here.

It is not difficult to see that there are exactly two maximal twin-free extensions of P_4 illustrated in Figure 4; their spectra are $2.56, 1, 0^2, -1.56, -2$ and $3, 1, 0^2, (-2)^2$. The path P_6 counts 11 maximal twin-free extensions having between 12 and 14 vertices, not listed here. For P_8 this number is 1639, and the extensions have between 18 and 30 vertices.

The half graph H_8 has 1756 maximal extensions, again with between 18 and 30 vertices.

Finally, C_6 has exactly 2 maximal extensions. If their adjacency matrices are as in Theorem 2.1, then the submatrices $(A_X\ B^\intercal)$ are

$$\begin{pmatrix} 0&0&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&1&1&1&1&1&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&1&0\\ 0&0&1&0&0&1&1&1&0&1&0&1&0\\ 0&0&1&1&1&0&1&0&0&1&1&0&1\\ 1&0&1&0&1&1&0&0&1&0&0&1&1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&1&0&0&0&1&0&0&0\\ 0&1&0&0&1&0&0&0&1&1&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&1&0&0&1&1&1&1&0&1&0&1&0\\ 0&1&0&0&1&0&0&1&0&1&0&1&1&0\\ 0&1&0&0&1&0&0&1&1&1&0&1&1&0\\ 0&0&0&1&1&1&1&0&1&0&0&1&0&1 \end{pmatrix}.$$

The spectra are $5.24, 1.79^2, 0^7, (-2.79)^2, -3.24$ and $5.68, 2, 1.70, 0^8, -2.25, -3.13, -4$, respectively.

To give an insight in complexity of these results, we mention that for C_{10} there exist 317 vertices with distinct C_{10}-neighborhoods satisfying item (i) of Theorem 2.5.

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