Non-Archimedean White Noise, Pseudodifferential Stochastic Equations, and Massive Euclidean Fields.

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Abstract

W. A. Zúñiga-Galindo, Non-Archimedean White Noise, Pseudodifferential Stochastic Equations of Klein-Gordon Type, and Massive Euclidean Fields, arXiv:1501.00707.

- We construct p-adic Euclidean random fields $\boldsymbol{\Phi}$ over $\mathbb{Q}_p^N$, for arbitrary N, these fields are solutions of p-adic stochastic pseudodifferential equations. From a mathematical perspective, the Euclidean fields are generalized stochastic processes parametrized by functions belonging to a nuclear countably Hilbert space, these spaces are introduced in this article, in addition, the Euclidean fields are invariant under the action of certain group of transformations. We also study the Schwinger functions of $\boldsymbol{\Phi}$.

Presentation Plan

- Notation
- Some comments on the Archimedean case
- A new class of non-Archimedean nuclear spaces
- Non-Archimedean white noise
- Euclidean random fields as convoluted generalized white noise
- Schwinger Functions
- Final comments

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- $\mathcal{D}'(\mathbb{Q}_p^N)$ denotes the space of distributions.

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- Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuos map from $\mathbb{Q}_p$ into the unit circle satisfying $\chi_p(y_0 + y_1) = \chi_p(y_0)\chi_p(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$.

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- Let $\mathfrak{B}(x,y)$ be a symmetric non-degenerate $\mathbb{Q}_p-$bilinear form on $\mathbb{Q}_p^N \times \mathbb{Q}_p^N$. Thus $\mathfrak{q}(x) := \mathfrak{B}(x,x)$, $x \in \mathbb{Q}_p^N$ is a *non-degenerate quadratic form* on $\mathbb{Q}_p^N$. We recall that

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- We identify the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^N$ with its algebraic dual $\left(\mathbb{Q}_p^N\right)^*$ by means of $\mathfrak{B}(\cdot, \cdot)$.

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- We now identify the dual group (i.e. the Pontryagin dual) of $\left(\mathbb{Q}_p^N, +\right)$ with $\left(\mathbb{Q}_p^N\right)^*$ by taking $x^*(x) = \chi_p\left(\mathfrak{B}\left(x, x^*\right)\right)$.

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- The Fourier transform is defined by

$$(\mathcal{F}g)(\xi) = \int_{\mathbb{Q}_p^N} g(x)\,\chi_p\left(\mathfrak{B}(x,\xi)\right) d\mu(x), \quad \text{for } g \in L^1,$$

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- The measure $d\mu\left(x\right)$ can be normalized uniquely in such manner that $(\mathcal{F}(\mathcal{F}g))(x) = g(-x)$ for every g belonging to $\mathcal{L}\left(\mathbb{Q}_p^N\right)$. Notice that $d\mu\left(x\right) = C(\mathfrak{q})d^N x$ where $C(\mathfrak{q})$ is a positive constant and $d^N x$ is the Haar measure on $\mathbb{Q}_p^N$ normalized by the condition $vol(B_0^N) = 1$.

Some comments on the Archimedean case

- A program of constructing Euclidean random fields of Markovian type by solving pseudo-stochastic partial differential equations of the form $LX = F$ with F a Euclidean noise and L a suitable invariant pseudodifferential operator was started in in the 70's by D. Surgailis and S. Albeverio and R. Høegh-Krohn, among others.

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- $\left(-\Delta + m^2\right)^{\alpha} X = F$ for $\alpha \in (0, 1)$ and $m \geq 0$.

- For $\alpha = \frac{1}{2}$, F the Gaussian white noise, X is the Nelson's Euclidean free field over $\mathbb{R}^d$.

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Albeverio, Sergio; Wu, Jiang Lun Euclidean random fields obtained by convolution from generalized white noise. J. Math. Phys. 36 (1995), no. 10, 5217–5245.

Albeverio, Sergio; Gottschalk, Hanno; Wu, Jiang-Lun Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions. Rev. Math. Phys. 8 (1996), no. 6, 763–817.

A true non-Archimedean analog of the Schwartz space

- The p-adic analog should have the form

$$\left(\mathbf{L}_{\alpha}+m^{2}\right)\boldsymbol{\Phi}=F,$$

where $\mathbf{L}_{\alpha}$ is a p-adic Laplacian (an elliptic pseudodifferential operator), the mass m is a positive real number, and $\boldsymbol{\Phi}\left(f,T\right)$ is the Euclidean quantum field (which is a generalized random process parametrized by functions f belonging to a suitable space X and $T\in X^{*}$, T plays the role of the ω.

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- Take $\mathbf{L}_{\alpha}=D^{\alpha}$ be the Vladimirov operator and $m>0$, then $\mathcal{D}\left(\mathbb{Q}_{p}\right)$ is not invariant under the action of $D^{\alpha}+m^{2}$!

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- We construct an space $\mathcal{H}_{\mathbb{C}}(\infty)$, which is nuclear and invariant under the action of a large class of pseudodifferential operators. In addition, it contains a dense copy of $\mathcal{D}(\mathbb{Q}_p^N)$.

The Green function of $\left(-\Delta+m^{2}\right)^{\alpha}$ is

$$\begin{aligned} G(x) &= \mathcal{F}^{-1}_{\xi\rightarrow x}\left(\frac{1}{\left[\left|\xi\right|^{2}+m^{2}\right]^{\alpha}}\right) \text{ in } \mathcal{S}'(\mathbb{R}^{N}) \\ &= \int\limits_{\mathbb{R}^{N}} \frac{\exp\left(-2\pi i\xi\cdot x\right)}{\left[\left|\xi\right|^{2}+m^{2}\right]^{\alpha}} d^{N}\xi \quad \textbf{(formally)}, \end{aligned}$$

where $\left|\xi\right|^{2}=\sum_{i=1}^{N}\xi_{i}^{2}$ and $\xi\cdot x=\sum_{i=1}^{N}\xi_{i}x_{i}$.

The quadratic form attache to $\xi\cdot x$ is exactly $\left|\xi\right|^{2}$.

Symmetries Archimedean vs. non-Archimedean

- If $g \in GL_N$ is a symmetry of $G(x)$, then $g^T E g = E$, where E is the identity matrix. Then the group of symmetries of $G(x)$ is $O(N) = \left\{g \in GL_N; g^T E g = E\right\}$ (because if g preserves $|\xi|^2$ then g preserves $\xi \cdot x$.)

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- $(\mathbf{L}_\gamma \boldsymbol{\varphi})(x) = \mathcal{F}^{-1}_{\xi \to x}(|l(\xi)|_p^\gamma \mathcal{F}_{x \to \xi}\varphi)$ where $\gamma > 0$ and $l(\xi) \in \mathbb{Q}_p[\xi_1, \cdots, \xi_N]$ satisfies $l(\xi) = 0 \Leftrightarrow \xi = 0$.

- $\mathbf{L}_\gamma$ is an elliptic operator.

Symmetries Archimedean vs. non-Archimedean

- In the case $N = 4$ there is a unique elliptic quadratic form, up to linear equivalence, which is $\mathfrak{l}_4(\xi) = \xi_1^2 - s\xi_2^2 - p\xi_3^2 + s\xi_4^2$, where $s \in \mathbb{Z} \smallsetminus \{0\}$ is a quadratic non-residue, i.e. $\left(\frac{s}{p}\right) = -1$.

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- The matrix of $\mathfrak{q}(\xi)$ is E and the matrix of $\mathfrak{l}_4(\xi)$ is $diag\,[1, -s, -p, s]$. If $\mathfrak{q}(\xi)$ is elliptic there is a non-singular matrix g such that

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- Which are the p-adic analogs of $\left(-\Delta + m^2\right)^{\alpha}$?

Symmetries Archimedean vs. non-Archimedean

- There are two choices:

$$\mathcal{F}^{-1}_{\xi\to x}\left(\left[\left|l(\xi)\right|_p^{\alpha}+m^2\right]\mathcal{F}_{x\to\xi}\varphi\right) \qquad (A)$$

$$\mathcal{F}^{-1}_{\xi\to x}\left(\left[\left|l(\xi)\right|_p+m^2\right]^{\alpha}\mathcal{F}_{x\to\xi}\varphi\right) \qquad (B).$$

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- $\left(\mathbf{L}_\alpha + m^2\right)\varphi = \mathcal{F}^{-1}_{\xi\to x}\left(\left[\left|l(\xi)\right|_p^\alpha + m^2\right]\mathcal{F}_{x\to\xi}\varphi\right).$

Symmetries Archimedean vs. non-Archimedean

- The Green function of $\mathbf{L}_\alpha + m^2$, using the classical definition for Fourier transform,

$$\begin{aligned} G(x) &= \mathcal{F}^{-1}_{\xi\to x}\left(\frac{1}{|l(\xi)|_p^\alpha + m^2}\right) \text{ in } \mathcal{D}'(\mathbb{Q}_p^N) \\ &= \int_{\mathbb{Q}_p^N} \frac{\chi_p(-\xi\cdot x)}{|l(\xi)|_p^\alpha + m^2} d^N\xi \quad \textbf{(formally)}, \end{aligned}$$

where $\xi \cdot x = \sum_{i=1}^N \xi_i x_i$.

Now there are two different forms: $\sum_{i=1}^N \xi_i^2$ and $l(\xi)$. The group of symmetries preserving simultaneously both forms can be trivial or very small.

Symmetries Archimedean vs. non-Archimedean

- Consider the case $N = 2$, $\mathfrak{q}(\xi) = \xi_1^2 + \xi_2^2$, the quadratic form attached to $\xi \cdot x = \sum_{i=1}^{2} \xi_i x_i$, and $l(\xi) = \xi_1^2 - \tau\xi_2^2$ with $\tau \in \mathbb{Q}_p \smallsetminus \{0\}$, $\tau \neq -1$, τ is not a square in $\mathbb{Q}_p \smallsetminus \{0\}$.

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- We are looking for transformations $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in GL_2(\mathbb{Q}_p)$ satisfying:

$$g^T E g = E \Leftrightarrow g^T g = E \Leftrightarrow g^T = g^{-1}$$

$$g^T \begin{bmatrix} 1 & 0 \\ 0 & -\tau \end{bmatrix} g = \begin{bmatrix} 1 & 0 \\ 0 & -\tau \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -\tau \end{bmatrix} g = g \begin{bmatrix} 1 & 0 \\ 0 & -\tau \end{bmatrix}$$

$$\begin{bmatrix} g_{11} & g_{12} \\ -\tau g_{21} & -\tau g_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & -\tau g_{12} \\ g_{21} & -\tau g_{22} \end{bmatrix}$$

$g_{12} = -\tau g_{12} \Leftrightarrow (1+\tau)\, g_{12} = 0 \Leftrightarrow g_{12} = 0$ because $\tau \neq -1$. Similarly $g_{21} = g_{22} = 0$, thus $g = \begin{bmatrix} g_{11} & 0 \\ 0 & 0 \end{bmatrix}$ which is a singular matrix. Hence in this case the group of symmetries is trivial.

For this reason we have to work with more general Fourier transforms, so that

$$\begin{aligned} G(x) &= \mathcal{F}^{-1}_{\xi \to x}\left(\frac{1}{|l(\xi)|_p^{\alpha} + m^2}\right) \text{ in } \mathcal{D}'(\mathbb{Q}_p^N) \\ &= \int_{\mathbb{Q}_p^N} \frac{\chi_p\left(-\mathfrak{B}\left(x,\xi\right)\right) d\mu\left(x\right)}{|l(\xi)|_p^{\alpha} + m^2} d^N\xi \quad \textbf{(formally)}, \end{aligned}$$

may have a chance of having a non-trivial group of symmetries. Of course, it is necessary that the quadratic form $\mathfrak{B}\left(\xi,\xi\right)$ and the form $l(\xi)$ are related nicely in order to have a non-trivial group of symmetries.

A true non-Archimedean analog of the Schwartz space

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- We define for f, g in $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ (or in $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$) the following scalar product:

$$\langle f, g \rangle_{l,\alpha} := \langle f, g \rangle_l = \int_{\mathbb{Q}_p^N} \left[\max\left(1, \|\xi\|_p\right) \right]^{2\alpha l} \widehat{f}(\xi) \overline{\widehat{g}}(\xi) \, d^N \xi, \qquad (2)$$

for a fixed $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and $l \in \mathbb{Z}$.

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for a fixed $\alpha \in \mathbb{R}_+ \smallsetminus \{0\}$ and $l \in \mathbb{Z}$.

- We also set $\|f\|_{l,\alpha}^2 =: \|f\|_l^2 = \langle f, f \rangle_l$. Notice that $\|\cdot\|_m \leq \|\cdot\|_n$ for $m \leq n$. Let denote by $\mathcal{H}_{\mathbb{R}}\left(\mathbb{Q}_p^N; l, \alpha\right) =: \mathcal{H}_{\mathbb{R}}(l)$ the completion of $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ with respect to $\langle \cdot, \cdot \rangle_l$.

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- Notice that $\mathcal{H}_{\mathbb{R}}(0) = L^2_{\mathbb{R}}$ and that $\mathcal{H}_{\mathbb{R}}(\infty) \subset L^2_{\mathbb{R}}$. With the topology induced by the family of seminorms $\|\cdot\|_{l \in \mathbb{N}}$, $\mathcal{H}_{\mathbb{R}}(\infty)$ becomes a locally convex space, which is metrizable. Indeed,

$$d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}$$

is a metric for the topology of $\mathcal{H}_{\mathbb{R}}(\infty)$ considered as a convex topological space.

A true non-Archimedean analog of the Schwartz space

Remark

We denote by $\mathcal{H}_{\mathbb{C}}(l)$, $\mathcal{H}_{\mathbb{C}}(\infty)$ *the* $\mathbb{C}$*-vector spaces constructed from* $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$. *All the above results are valid for these spaces. We shall use d to denote the metric of* $\mathcal{H}_{\mathbb{C}}(\infty)$.

Lemma

$\mathcal{H}_{\mathbb{R}}(\infty)$ *endowed with the topology* τ_P *is a countably Hilbert space in the sense of Gel'fand and Vilenkin. Furthermore* $(\mathcal{H}_{\mathbb{R}}(\infty), \tau_P)$ *is metrizable and complete and hence a Fréchet space.*

A true non-Archimedean analog of the Schwartz space

Lemma

(i) Set $\overline{\left(\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N), d\right)}$ *for the completion of the metric space* $\left(\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N), d\right)$*. Then* $\overline{\left(\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N), d\right)} = \left(\mathcal{H}_{\mathbb{R}}\left(\infty\right), d\right)$*.*
(ii) $\left(\mathcal{H}_{\mathbb{R}}\left(\infty\right), d\right)$ *is a nuclear space.*

Proof.

(i) it is established by an argument based on sequences.
(ii) We recall that $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$ is a nuclear space, and thus $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ is a nuclear space, since any subspace of a nuclear space is also nuclear. Now, since the completion of a nuclear space is also nuclear, by (i), $\mathcal{H}_{\mathbb{R}}\left(\infty\right)$ is a nuclear space. □

A true non-Archimedean analog of the Schwartz space

Theorem

$\mathcal{H}_{\mathbb{R}}(\infty)$ is nuclear countably Hilbert space.

Remark

(i) As a nuclear Fréchet space $\mathcal{H}_{\mathbb{R}}(\infty)$ admits a sequence of defining Hilbertian norms $|\cdot|_{m\in\mathbb{N}}$ such that (1) $|g|_m \leq C_m |g|_{m+1}$, $g \in \mathcal{H}_{\mathbb{R}}(\infty)$, with some $C_m > 0$; (2) the canonical map $i_{n,n+1} : H_{\mathbb{R}}(n+1) \longrightarrow H_{\mathbb{R}}(n)$ is of Hilbert-Schmidt type, where $H_{\mathbb{R}}(n)$ is the Hilbert space associated with $|\cdot|_n$.
(ii) Let $\mathcal{H}^*_{\mathbb{R}}(l)$ be the dual space of $\mathcal{H}_{\mathbb{R}}(l)$. By identifying $\mathcal{H}^*_{\mathbb{R}}(l)$ with $\mathcal{H}_{\mathbb{R}}(-l)$ and denoting the dual pairing between $\mathcal{H}^*_{\mathbb{R}}(\infty)$ and $\mathcal{H}_{\mathbb{R}}(\infty)$ by $\langle\cdot,\cdot\rangle$, we have from the results of Gel'fand and Vilenkin that $\mathcal{H}^*_{\mathbb{R}}(\infty) = \cup_{l\in\mathbb{N}}\mathcal{H}^*_{\mathbb{R}}(l)$. We shall consider $\mathcal{H}^*_{\mathbb{R}}(\infty)$ as equipped with the weak topology.

A true non-Archimedean analog of the Schwartz space

Lemma

For any $l \in \mathbb{N}$, *we set*

$$d\nu_{l,N} := \left[\max\left(1, \|\xi\|_p\right)\right]^{2\alpha l} d^N\xi,$$

and

$$L^2_{l,N} := \left\{ f : \mathbb{Q}_p^N \to \mathbb{C} : \int_{\mathbb{Q}_p^N} \left|\widehat{f}\right|^2 d\nu_{l,N} < \infty \right\}.$$

Notice that $L^2_{l,N} \subset L^2$. *Then* $\mathcal{H}_{\mathbb{C}}(l) = L^2_{l,N}$ *for any* $l \in \mathbb{N}$. *A similar result is valid for* $\mathcal{H}_{\mathbb{R}}(l)$.

Pseudodifferential operators

Definition

We say that a function $\mathfrak{a} : \mathbb{Q}_p^N \rightarrow \mathbb{R}_+$ is a smooth symbol, if it satisfies the following properties:
(i) $\mathfrak{a}$ is a continuous function;
(ii) there exists a positive constant $C = C(\mathfrak{a})$ such that $\mathfrak{a}(\xi) \geq C$ for any $\xi \in \mathbb{Q}_p^N$;
(iii) there exist positive constants C_0, C_1, α, m_0, with $m_0 \in \mathbb{N}$, such that

$$C_0 \|\xi\|_p^{\alpha} \leq \mathfrak{a}(\xi) \leq C_1 \|\xi\|_p^{\alpha} \quad \text{for} \quad \|\xi\|_p \geq p^{m_0}.$$

Pseudodifferential operators

Given a smooth symbol $\mathfrak{a}(\xi)$, we attach to it the following pseudodifferential operator:

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N) & \rightarrow & L^2 \cap C^{\text{unif}} \\ g & \rightarrow & \mathbf{A}g, \end{array}$$

where $(\mathbf{A}g)(x) = \mathcal{F}^{-1}_{\xi \rightarrow x}\left(\mathfrak{a}(\xi)\,\mathcal{F}_{x\rightarrow\xi} g\right)$.

Example

$\left(\mathbf{L}_\alpha + m^2\right)$

Lemma

For any $l \in \mathbb{N}$, the mapping $\mathbf{A} : \mathcal{H}_{\mathbb{C}}(l+1) \rightarrow \mathcal{H}_{\mathbb{C}}(l)$ is a well-defined continuous mapping between Banach spaces.

The above lemma is the non-Archimedean counterpart of the following fact:

$$\begin{array}{ccc} C^{r+1}(\mathbb{R}) & \longrightarrow & C^{r}(\mathbb{R}) \\ f & \longrightarrow & \frac{d}{dx}f. \end{array}$$

Theorem

(i) The mapping $\mathbf{A}:\mathcal{H}_{\mathbb{C}}(\infty)\rightarrow\mathcal{H}_{\mathbb{C}}(\infty)$ *is a bi-continuous isomorphism of locally convex spaces. (ii)* $\mathcal{H}_{\mathbb{C}}(\infty)\subset L^{\infty}\cap C^{unif}\cap L^{1}\cap L^{2}$.

Pseudodifferential Operators and Green Functions

- We take $\mathfrak{l}(\xi) \in \mathbb{Z}_p[\xi_1, \cdots, \xi_N]$ to be an *elliptic polynomial* of degree d, this means that $\mathfrak{l}$ is homogeneous of degree d and satisfies $\mathfrak{l}(\xi) = 0 \Leftrightarrow \xi = 0$. There are infinitely many elliptic polynomials.

Pseudodifferential Operators and Green Functions

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- We consider the following *elliptic pseudodifferential operator*:

$$(\mathbf{L}_\alpha h)(x) = \mathcal{F}^{-1}_{\xi \to x}\left(|\mathfrak{l}(\xi)|_p^\alpha \mathcal{F}_{x\to\xi} h\right),$$

where $\alpha > 0$ and $h \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$.

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where $\alpha > 0$ and $h \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$.

- We shall call a fundamental solution $G(x; m, \alpha)$ of the equation

$$\left(\mathbf{L}_\alpha + m^2\right) u = h, \text{ with } h \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N), \ m > 0, \tag{3}$$

a *Green function* of $\mathbf{L}_\alpha$.

Pseudodifferential Operators and Green Functions

- As a distribution on $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$, the Green function is given by

$$G(x; m, \alpha) = \mathcal{F}^{-1}_{\xi \to x}\left(\frac{1}{\left| \mathfrak{l}(\xi) \right|_p^{\alpha} + m^2} \right). \tag{4}$$

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- Notice that since

$$C_0^{\alpha}\left\|\xi\right\|_p^{\alpha d}\leq\left|\mathfrak{l}\left(\xi\right)\right|^{\alpha}\leq C_1^{\alpha}\left\|\xi\right\|_p^{\alpha d}, \tag{5}$$

for some positive constants C_0, C_1,

$$\frac{1}{\left|\mathfrak{l}\left(\xi\right)\right|_p^{\alpha}+m^2}\in L^1\left(\mathbb{Q}_p^N,d^N\xi\right)\quad\text{for}\quad\alpha d>N,$$

and in this case, $G\left(x;m,\alpha\right)$ is an L^{∞}-function.

Pseudodifferential Operators and Green Functions

Proposition

The Green function $G(x; m, \alpha)$ *verifies the following properties:*
(i) the function $G(x; m, \alpha)$ *is continuous on* $\mathbb{Q}_p^N \smallsetminus \{0\}$*;*
(ii) if $\alpha d > N$*, then the function* $G(x; m, \alpha)$ *is continuous;*
(iii) for $0 < \alpha d \leq N$*, the function* $G(x; m, \alpha)$ *is locally constant on* $\mathbb{Q}_p^N \smallsetminus \{0\}$*, and*

$$|G(x; m, \alpha)| \leq \begin{cases} C \|x\|_p^{2\alpha d - N} & \text{for} \quad 0 < \alpha d < N \\ C_0 - C_1 \ln \|x\|_p & \text{for} \quad N = \alpha d, \end{cases}$$

for $\|x\|_p \leq 1$*, where* C*,* C_0*,* C_1 *are positive constants,*
(iv) $|G(x; m, \alpha)| \leq C_2 \|x\|_p^{-\alpha d - N}$ *as* $\|x\|_p \to \infty$*, where* C_2 *is positive constant;*
(v) $G(x; m, \alpha) \geq 0$ *on* $\mathbb{Q}_p^N \smallsetminus \{0\}$*.*

Pseudodifferential Operators and Green Functions

Kochubei, Anatoly N. Pseudo-differential equations and stochastics over non-Archimedean fields. Monographs and Textbooks in Pure and Applied Mathematics, 244. Marcel Dekker, Inc., New York, 2001.

Theorem

Let $\alpha > 0$, $m > 0$, and let $\mathbf{L}_\alpha$ be an elliptic operator. (i) There exists a Green function $G\left(x; m, \alpha\right)$ for the operator $\mathbf{L}_\alpha$, which is continuous and non-negative on $\mathbb{Q}_p^n \smallsetminus \{0\}$, and tends to zero at infinity. Furthermore, if $h \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$, then $u(x) = G\left(x; m, \alpha\right) * h(x)$ is a solution of (3) in $\mathcal{D}_{\mathbb{C}}^*(\mathbb{Q}_p^N)$. (ii) The equation

$$\left(\mathbf{L}_\alpha + m^2\right) u = g, \tag{6}$$

with $g \in \mathcal{H}_{\mathbb{R}}\left(\infty\right)$, has a unique solution $u \in \mathcal{H}_{\mathbb{R}}\left(\infty\right)$.

Proof.

(ii) By a density argument we can take $g \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$, then by (i), $u(x) = G(x; m, \alpha) * g(x)$ is a real-valued, locally constant function which is a solution of (6) in $\mathcal{D}_{\mathbb{C}}^{*}(\mathbb{Q}_p^N)$. Now, since $\widehat{u}(\xi) = \frac{\widehat{g}(\xi)}{|\mathfrak{l}(\xi)|_p^{\alpha} + m^2} \in L^2$,

$$\|u\|_{l+d}^2 \leq C\|g\|_0^2 + \int\limits_{\mathbb{Q}_p^N \smallsetminus B_0^N} \frac{\|\xi\|_p^{2\alpha(l+d)} |\widehat{g}(\xi)|^2 d^N\xi}{\left(|\mathfrak{l}(\xi)|_p^{\alpha} + m^2\right)^2}$$

$$\leq C\|g\|_0^2 + \frac{1}{C_0^{\alpha}} \int\limits_{\mathbb{Q}_p^N \smallsetminus B_0^N} \|\xi\|_p^{2\alpha l} |\widehat{g}(\xi)|^2 d^N\xi \leq C\|g\|_0^2 + \frac{1}{C_0^{\alpha}}\|g\|_l^2$$

$$\leq C'\|g\|_l^2, \text{ for } l \in \mathbb{N}.$$

Then, by Lemma 4, $u \in \mathcal{H}_{\mathbb{R}}(m)$, for $m \geq d$. In the case, $0 \leq m \leq d-1$, one gets $\|u\|_m \leq C''\|g\|_0$. Therefore $u \in \mathcal{H}_{\mathbb{R}}(m)$, for $m \in \mathbb{N}$. □

Pseudodifferential Operators and Green Functions

Corollary

The mapping

$$\begin{array}{ccc} \mathcal{H}_{\mathbb{R}}(\infty) & \rightarrow & \mathcal{H}_{\mathbb{R}}(\infty) \\ g(x) & \rightarrow & G(x;m,\alpha) * g(x), \end{array}$$

is continuous.

Notice that

$$G(x;m,\alpha) * g(x) = \left(\mathbf{L}_{\alpha} + m^2\right)^{-1} g.$$

Infinitely divisible probability distributions

- An infinitely divisible probability distribution P is a probability distribution having the property that for each $n \in \mathbb{N}$ there exists a probability distribution P_n such that $P = P_n * \cdots * P_n$ (n-times).

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- By the Lévy-Khinchine Theorem, the characteristic function C_P of P satisfies

$$C_P(t) = \int_{\mathbb{R}} e^{ist} dP(s) = e^{\Psi(t)},\ t \in \mathbb{R}, \tag{7}$$

where $\Psi : \mathbb{R} \to \mathbb{C}$ is a continuous function, called the *Lévy characteristic of* P, which is uniquely represented as follows:

$$\Psi(t) = iat - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\smallsetminus\{0\}} \left(e^{ist} - 1 - \frac{ist}{1+s^2} \right) dM(s),\ t \in \mathbb{R}, \tag{8}$$

where a, $\sigma \in \mathbb{R}$, with $\sigma \geq 0$, and the measure $dM(s)$ satisfies

$$\int_{\mathbb{R}\smallsetminus\{0\}} \min\left(1, s^2\right) dM(s) < \infty. \tag{9}$$

Infinitely divisible probability distributions

Remark

From now on, we work with infinitely divisible probability distributions which are absolutely continuous with all finite moments. This fact is equivalent to all the moments of the corresponding M's are finite, cf. [1, Theorem 2.3].

Albeverio Sergio, Wu Jiang Lun, Euclidean random fields obtained by convolution from generalized white noise, J. Math. Phys. 36 (1995), no. 10, 5217–5245.

Bochner-Minlos Theorem

- Let $\mathcal{H}_{\mathbb{R}}(\infty)$ and $\mathcal{H}^{*}_{\mathbb{R}}(\infty)$ be the spaces introduced before. We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $\mathcal{H}_{\mathbb{R}}(\infty)$ and $\mathcal{H}^{*}_{\mathbb{R}}(\infty)$. Let $\mathcal{B}$ be the σ-algebra generated by cylinder sets of $\mathcal{H}^{*}_{\mathbb{R}}(\infty)$. Then $\left(\mathcal{H}^{*}_{\mathbb{R}}(\infty), \mathcal{B}\right)$ is a measurable space.

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- (i) C is continuous on $\mathcal{H}_{\mathbb{R}}(\infty)$;
- (ii) C is positive-definite;
- (iii) $C(0) = 1$.

Bochner-Minlos Theorem

Since $\mathcal{H}_{\mathbb{R}}(\infty)$ is a nuclear space, cf. Theorem 3, by the Bochner-Minlos Theorem, there exists a one to one correspondence between the characteristic functionals C and probability measures P on $\left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}\right)$ given by the following relation

$$C(f) = \int\limits_{\mathcal{H}_{\mathbb{R}}^{*}(\infty)} e^{i\langle f, T\rangle} \mathrm{dP}(T), \ f \in \mathcal{H}_{\mathbb{R}}(\infty).$$

The generalized white noise

Theorem

Let Ψ be a Lévy characteristic defined by (7). Then there exists a unique probability measure P_Ψ on $\left(\mathcal{H}^*_{\mathbb{R}}(\infty), \mathcal{B}\right)$ such that the Fourier transform of P_Ψ satisfies

$$\int_{\mathcal{H}^*_{\mathbb{R}}(\infty)} e^{i\langle f,T\rangle} \mathrm{d}P_\Psi(T) = \exp\left\{\int_{\mathbb{Q}_p^N} \Psi(f(x))\, d^N x\right\},\ f \in \mathcal{H}_{\mathbb{R}}(\infty).$$

The proof is based on [1, Theorem 6, p. 283] like in the Archimedean case, cf. [1, Theorem 1.1]. However, in the non-Archimedean case the result does not follow directly from [1]. We need some additional results.

Gel'fand I. M., Vilenkin N. Ya, Generalized functions. Vol. 4. Applications of harmonic analysis. Academic Press, New York-London, 1964.

The generalized white noise

Lemma

$\int_{\mathbb{Q}_p^N} \Psi\left(f\left(x\right)\right) d^N x < \infty$ for any $f \in \mathcal{H}_{\mathbb{R}}\left(\infty\right)$.

Lemma

The function $f \rightarrow \int_{\mathbb{Q}_p^N} \Psi\left(f\left(x\right)\right) d^N x$ is continuous on $\mathcal{H}_{\mathbb{R}}\left(\infty\right)$.

Set $L(f) := \exp\left\{\int_{\mathbb{Q}_p^N} \Psi\left(f\left(x\right)\right) d^N x\right\}$ for $f \in \mathcal{H}_{\mathbb{R}}\left(\infty\right)$. Notice that by Lemma 13 this function is well-defined.

Proposition

The function $L(f)$ is positive-definite if and only if $e^{s\Psi(t)}$ is positive-definite for every $s > 0$.

Non-Archimedean generalized white noise measures

Definition

We call P_{Ψ} in Theorem 12 a generalized white noise measure with Lévy characteristic Ψ and $\left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}, \mathrm{P}_{\Psi}\right)$ the generalized white noise space associated with Ψ. The associated coordinate process

$$F : \mathcal{H}_{\mathbb{R}}(\infty) \times \left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}, \mathrm{P}_{\Psi}\right) \to \mathbb{R}$$

defined by $F(f, T) = \langle f, T \rangle$, $f \in \mathcal{H}_{\mathbb{R}}(\infty)$, $T \in \mathcal{H}_{\mathbb{R}}^{*}(\infty)$, is called generalized white noise.

The generalized white noise F is composed by three independent noises: constant, Gaussian and Poisson (with jumps given by M) noises.

Euclidean random fields as convoluted generalized white noise

Definition

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. By a generalized random field $\mathbf{\Phi}$ on $(\Omega, \mathcal{F}, P)$ with parameter space $\mathcal{H}_{\mathbb{R}}(\infty)$, we mean a system

$$\left\{\mathbf{\Phi}(g, \omega) : \omega \in \Omega\right\}_{g \in \mathcal{H}_{\mathbb{R}}(\infty)},$$

of random variables on $(\Omega, \mathcal{F}, P)$ having the following properties:
(i) $P\left\{\omega \in \Omega : \mathbf{\Phi}(c_1 g_1 + c_2 g_2, \omega) = c_1 \mathbf{\Phi}(g_1, \omega) + c_2 \mathbf{\Phi}(g_2, \omega)\right\} = 1$, for $c_1, c_2 \in \mathbb{R}$, $g_1, g_2 \in \mathcal{H}_{\mathbb{R}}(\infty)$;
(ii) if $g_n \to g$ in $\mathcal{H}_{\mathbb{R}}(\infty)$, then $\mathbf{\Phi}(g_n, \omega) \to \mathbf{\Phi}(g, \omega)$ in law.

The coordinate process in Definition 16 is a random field on the generalized white noise space $\left(\mathcal{H}^*_{\mathbb{R}}(\infty), \mathcal{B}, \mathrm{P}_\Psi\right)$

Euclidean random fields as convoluted generalized white noise

- We now recall that $(\mathcal{G}f)(x) := G(x; m, \alpha) * f(x)$ gives rise to a continuous mapping from $\mathcal{H}_{\mathbb{R}}(\infty)$ into itself, cf. Corollary 11.

Euclidean random fields as convoluted generalized white noise

- We now recall that $(\mathcal{G}f)(x) := G(x; m, \alpha) * f(x)$ gives rise to a continuous mapping from $\mathcal{H}_{\mathbb{R}}(\infty)$ into itself, cf. Corollary 11.

- Thus, the conjugate operator $\widetilde{\mathcal{G}} : \mathcal{H}_{\mathbb{R}}^{*}(\infty) \rightarrow \mathcal{H}_{\mathbb{R}}^{*}(\infty)$ is a measurable mapping from $\left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}\right)$ into itself.

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- The generalized white noise measure P_{Ψ} on $\left(\mathcal{H}^{*}_{\mathbb{R}}(\infty), \mathcal{B}\right)$ associated with a Lévy characteristic Ψ was introduced in Definition 16.

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- The generalized white noise measure P_{Ψ} on $\left(\mathcal{H}^{*}_{\mathbb{R}}(\infty), \mathcal{B}\right)$ associated with a Lévy characteristic Ψ was introduced in Definition 16.

- We set $\mathrm{P}_{\mathbf{\Phi}}$ to be the image probability measure of P_{Ψ} under $\widetilde{\mathcal{G}}$, i.e. $\mathrm{P}_{\mathbf{\Phi}}$ is the measure on $\left(\mathcal{H}^{*}_{\mathbb{R}}(\infty), \mathcal{B}\right)$ defined by

$$\mathrm{P}_{\mathbf{\Phi}}(A) = \mathrm{P}_{\Psi}\left(\widetilde{\mathcal{G}}^{-1}(A)\right), \text{ for } A \in \mathcal{B}. \qquad (10)$$

Euclidean random fields as convoluted generalized white noise

Proposition

The Fourier transform of $\mathrm{P}_{\mathbf{\Phi}}$ *is given by*

$$\int_{\mathcal{H}_{\mathbb{R}}^{*}(\infty)} e^{i\langle f,T\rangle} d\mathrm{P}_{\mathbf{\Phi}}(T) = \exp\left\{ \int_{\mathbb{Q}_p^N} \Psi\left\{ \int_{\mathbb{Q}_p^N} G(x-y;m,\alpha)\, f(y)\, d^N y \right\} d^N x \right\},$$

for $f \in \mathcal{H}_{\mathbb{R}}(\infty)$.

Euclidean random fields as convoluted generalized white noise

- By Proposition 18, the associated coordinate process

$$\mathbf{\Phi} : \mathcal{H}_{\mathbb{R}}(\infty) \times \left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}\right) \rightarrow \mathbb{R}$$

given by $\mathbf{\Phi}(f, T) = \langle \mathcal{G}f, T \rangle$, $f \in \mathcal{H}_{\mathbb{R}}(\infty)$, $T \in \left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}\right)$, is a random field on $\left(\mathcal{H}_{\mathbb{R}}^{*}(\infty), \mathcal{B}, \mathrm{P}_{\mathbf{\Phi}}\right)$.

Euclidean random fields as convoluted generalized white noise

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- In fact, $\mathbf{\Phi}$ is nothing but $\widetilde{\mathcal{G}}F$ which is defined by

$$\widetilde{\mathcal{G}}F(f, T) = F(\mathcal{G}f, T), \; f \in \mathcal{H}_{\mathbb{R}}(\infty), \; T \in \mathcal{H}_{\mathbb{R}}^{*}(\infty).$$

Euclidean random fields as convoluted generalized white noise

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given by $\mathbf{\Phi}\left(f,T\right)=\left\langle \mathcal{G}f,T\right\rangle$, $f\in\mathcal{H}_{\mathbb{R}}\left(\infty\right)$, $T\in\left(\mathcal{H}_{\mathbb{R}}^{*}\left(\infty\right),\mathcal{B}\right)$, is a random field on $\left(\mathcal{H}_{\mathbb{R}}^{*}\left(\infty\right),\mathcal{B},\mathrm{P}_{\mathbf{\Phi}}\right)$.

- In fact, $\mathbf{\Phi}$ is nothing but $\widetilde{\mathcal{G}}F$ which is defined by

$$\widetilde{\mathcal{G}}F\left(f,T\right)=F\left(\mathcal{G}f,T\right),\ f\in\mathcal{H}_{\mathbb{R}}\left(\infty\right),\ T\in\mathcal{H}_{\mathbb{R}}^{*}\left(\infty\right).$$

- It is useful to see $\mathbf{\Phi}$ as the unique solution, in law, of the stochastic equation $\left(\mathbf{L}_{\alpha}+m^{2}\right)\mathbf{\Phi}=F$, where $\left(\mathbf{L}_{\alpha}+m^{2}\right)\mathbf{\Phi}\left(f,T\right):=\mathbf{\Phi}\left(\left(\mathbf{L}_{\alpha}+m^{2}\right)f,T\right)$, for $f\in\mathcal{H}_{\mathbb{R}}\left(\infty\right)$, $T\in\mathcal{H}_{\mathbb{R}}^{*}\left(\infty\right)$.

Symmetries

Given a polynomial $\mathfrak{a}(\xi) \in \mathbb{Q}_p[\xi_1, \cdots, \xi_n]$ and $\mathbf{g} \in GL_N(\mathbb{Q}_p)$, we say that $\mathbf{g}$ *preserves* $\mathfrak{a}$ if $\mathfrak{a}(\xi) = \mathfrak{a}(\mathbf{g}\xi)$, for all $\xi \in \mathbb{Q}_p^N$. By simplicity, we use $\mathbf{g}x$ to mean $[g_{ij}]\, x^T$, $x = (x_1, \cdots, x_N) \in \mathbb{Q}_p^N$, where we identify $\mathbf{g}$ with the matrix $[g_{ij}]$.

Definition

Let $\mathfrak{q}(\xi)$ be the elliptic quadratic form used in the definition of the Fourier transform, and let $\mathfrak{l}(\xi)$ be the elliptic polynomial that appears in the symbol of the operator $\mathbf{L}_\alpha$. We define the homogeneous Euclidean group of $\mathbb{Q}_p^N$ relative to $\mathfrak{q}(\xi)$ and $\mathfrak{l}(\xi)$, denoted as $E_0\left(\mathbb{Q}_p^N; \mathfrak{q}, \mathfrak{l}\right) := E_0\left(\mathbb{Q}_p^N\right)$, as the subgroup of $GL_N(\mathbb{Q}_p)$ whose elements preserve $\mathfrak{q}(\xi)$ and $\mathfrak{l}(\xi)$ simultaneously. We define the inhomogeneous Euclidean group, denoted as $E\left(\mathbb{Q}_p^N; \mathfrak{q}, \mathfrak{l}\right) := E\left(\mathbb{Q}_p^N\right)$, to be the group of transformations of the form $(a, \mathbf{g})\, x = a + \mathbf{g}x$, for $a, x \in \mathbb{Q}_p^N$, $\mathbf{g} \in E_0\left(\mathbb{Q}_p^N\right)$.

Symmetries

Let $(a, \mathbf{g})$ be a transformation in $E\left(\mathbb{Q}_p^N\right)$, the action of $(a, \mathbf{g})$ on a function $f \in \mathcal{H}_{\mathbb{R}}(\infty)$ is defined by

$$((a, \mathbf{g}) f)(x) = f\left((a, \mathbf{g})^{-1} x\right), \text{ for } x \in \mathbb{Q}_p^N,$$

and on a functional $T \in \mathcal{H}_{\mathbb{R}}^{*}(\infty)$, by

$$\langle f, (a, \mathbf{g}) T \rangle := \left\langle (a, \mathbf{g})^{-1} f, T \right\rangle, \text{ for } f \in \mathcal{H}_{\mathbb{R}}(\infty).$$

The action on a random field $\mathbf{\Phi}$ is defined by

$$((a, \mathbf{g}) \mathbf{\Phi})(f, T) = \mathbf{\Phi}\left((a, \mathbf{g})^{-1} f, T\right), \text{ for } f \in \mathcal{H}_{\mathbb{R}}(\infty), \ T \in \mathcal{H}_{\mathbb{R}}^{*}(\infty).$$

Symmetries

Example

In the case $N = 4$ there is a unique elliptic quadratic form, up to linear equivalence, which is $\mathfrak{l}_4(\xi) = \xi_1^2 - s\xi_2^2 - p\xi_3^2 + s\xi_4^2$, where $s \in \mathbb{Z} \setminus \{0\}$ is a quadratic non-residue, i.e. $\left(\frac{s}{p}\right) = -1$. We take $\mathfrak{q}_4(\xi) = \mathfrak{l}_4(\xi)$, i.e. $\mathfrak{B}_4(x, \xi) = \xi_1 x_1 - s\xi_2 x_2 - p\xi_3 x_3 + s\xi_4 x_4$. In this case, $E_0\left(\mathbb{Q}_p^4, \mathfrak{l}_4, \mathfrak{l}_4\right)$ equals

$$O(\mathfrak{l}_4) = \left\{ \mathbf{g} \in GL_4(\mathbb{Q}_p) : \mathbf{g}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \right\}$$

the orthogonal group of $\mathfrak{l}_4$.

Symmetries

Definition

By Euclidean invariance of the random field $\mathbf{\Phi}$ we mean that the laws of $\mathbf{\Phi}$ and $(a, \mathbf{g})\,\mathbf{\Phi}$ are the same for each $(a, \mathbf{g}) \in E\left(\mathbb{Q}_p^N\right)$, i.e. the probability distributions of $\left\{\mathbf{\Phi}\left(f, \cdot\right) : f \in \mathcal{H}_{\mathbb{R}}\left(\infty\right)\right\}$ and $\left\{\left(\left(a, \mathbf{g}\right)\mathbf{\Phi}\right)\left(f, \cdot\right) : f \in \mathcal{H}_{\mathbb{R}}\left(\infty\right)\right\}$ coincide for each $(a, \mathbf{g}) \in E\left(\mathbb{Q}_p^N\right)$.

We say that $\mathcal{G}$ is $(a, \mathbf{g})$-invariant for some $(a, \mathbf{g}) \in E\left(\mathbb{Q}_p^N\right)$, if $(a, \mathbf{g})\,\mathcal{G} = \mathcal{G}\,(a, \mathbf{g})$. If $\mathcal{G}$ is invariant under all $(a, \mathbf{g}) \in E\left(\mathbb{Q}_p^N\right)$, we say that $\mathcal{G}$ is *Euclidean invariant.*

Symmetries

Proposition

The random field $\mathbf{\Phi} = \widetilde{\mathcal{G}}_F$ is Euclidean invariant.

Schwinger Functions

Definition

Set $g_1, \cdots, g_m \in \mathcal{H}_{\mathbb{R}}(\infty)$. We define the $m-th$ Schwinger function S_m of $\mathbf{\Phi}$ as the $m-th$ moment of $\mathbf{\Phi}$, i.e.

$$S_m (g_1 \otimes \cdots \otimes g_m) = \int_{\mathcal{H}^*_{\mathbb{R}}(\infty)} \langle g_1, T\rangle \cdots \langle g_m, T\rangle \, d\mathrm{P}_{\mathbf{\Phi}}(T), \; m \in \mathbb{N} \smallsetminus \{0\}, \tag{11}$$

with $S_0 := 1$.

Schwinger Functions

Theorem

The Schwinger functions S_m *defined above are symmetric and Euclidean invariant functionals in* $\mathcal{H}^*_{\mathbb{R}}\left(\mathbb{Q}_p^{Nm};\infty\right)$ *for* $m \geq 1$*. Furthermore for* $g_1, \cdots, g_m \in \mathcal{H}_{\mathbb{R}}(\mathbb{Q}_p^N, \infty)$ *we have*

$$S_m\left(g_1 \otimes \cdots \otimes g_m\right) = \sum_{I \in P^{(m)}} \prod_{\{j_1, \cdots, j_l\} \in I} c_l \int_{\mathbb{Q}_p^N} \prod_{k=1}^{l} G\left(x; m, \alpha\right) * g_{j_k}\left(x\right) d^N x,$$

where

$$c_1 := a + \int_{\mathbb{R} \smallsetminus \{0\}} \tfrac{s^3}{1+s^2} dM\left(s\right), \qquad c_2 := \sigma^2 + \int_{\mathbb{R} \smallsetminus \{0\}} s^2 dM\left(s\right),$$

$$c_m := \int_{\mathbb{R} \smallsetminus \{0\}} s^m dM\left(s\right), \qquad \text{for } m \geq 3.$$