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Memorphic solutions of some difference equations in p-adic field

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1 -Introduction and preliminary results

Let K be an algebraically closed field, complete for an ultrametric absolute value. We denote by $\mathcal{A}(K)$ the K-algebra of entire functions in K and by $\mathcal{M}(K)$ the field of meromorphic functions in K, i.e. the field of fractions of $\mathcal{A}(K)$
This paper is devoted to the study of the growth of meromorphic solutions $y = f(x)$ of the functional equation :

$$(E) \qquad \sum_{i=0}^{s} g_i(x)y(q^i x) = h(x),$$

where $q \in K$ is such that $0 < |q| < 1$
and $h(x), g_0(x), \ ..., g_s(x), (s \geq 1)$ are meromorphic functions such that $g_0(x)g_s(x) \not\equiv 0$.

1 -Introduction and preliminary results

Indeed, recently many papers (see for example [3], [6], [7]) focused on such equations and many meaningful results have been obtained about the growth of their solution. Here we extend some of these results.
We also make a similar study of difference equations of the type :

$$\sum_{i=0}^{s} g_i(x)y(x+i) = h(x).$$

Throughout this paper, we use the ultrametric Nevanlinna theory (see e.g., [4], [5]). So we first have to recall some basic notions of this theory.

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Given $R > 0$, we denote by $D^-(0, R)$ the open disk $\{x \in K : |x| < R\}$ and by $D(0, R)$ the closed disk $\{x \in K : |x| \leq R\}$.
Similarly, we denote by $\mathcal{A}(D^-(0, R))$ the K-algebra of analytic functions inside the disk $D^-(0, R)$, i.e. the set of power series converging inside $D^-(0, R)$ and by $\mathcal{M}(D^-(0, R))$ the field of meromorphic functions in $D^-(0, R)$ i.e. the field of fractions of $\mathcal{A}(D^-(0, R))$.

For every $r \in]0, R[$, we define a multiplicative norm $|\ |(r)$ on $\mathcal{A}(D^-(0, R))$ by $|f|(r) = \sup_{n\geq 0} |a_n| r^n$ for every function $f(x) = \sum_{n\geq 0} a_n x^n$ of $\mathcal{A}(D^-(0, R))$.

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This norm is extended to $\mathcal{M}(D^-(0,R))$ as follows :
if $f \in \mathcal{M}(D^-(0,R))$ is given by $f = \frac{g}{h}$, with $g, h \in \mathcal{A}(D^-(0,R))$ we write :

$$|f|(r) = |\frac{g}{h}|(r) = \frac{|g|(r)}{|h|(r)}.$$

Finally, for every $f \in \mathcal{M}(D^-(0,R)) \setminus \{0\}$ and every $\alpha \in D^-(0,R)$ we denote by $\omega_\alpha(f)$ the integer i_α of $\mathbb{Z}$ such that $f(x) = \sum_{i \geq i_\alpha} a_i (x-\alpha)^i$ and $a_{i_\alpha} \neq 0$.

1 -Introduction and preliminary results

prposition 1.1

Let $R > 0$ and let $f \in \mathcal{M}(D^{-}(0,R))$ be such that 0 is neither a zero nor a pole of f. Then, for every $r \in]0,R[$, we have

$$\log|f|(r) = \log|f(0)| + \sum_{|\alpha| \leq r} \omega_{\alpha}(f) \log \frac{r}{|\alpha|}.$$

Let $f \in \mathcal{M}(D^{-}(0,R))$ be such that 0 is neither a zero nor a pole of f. For every $r \in]0,R[$, we denote by $Z(r,f)$ the counting function of zeros of f in the disk $D(0,r)$, counting multiplicity, i.e., we set

$$Z(r,f) = \sum_{\substack{\omega_{\alpha}(f)>0 \\ |\alpha| \leq r}} \omega_{\alpha}(f) \log \frac{r}{|\alpha|}.$$

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In the same way, we set

$$N(r,f) = Z(r,\frac{1}{f}),$$

to denote the counting function of poles of f in $D(0,r)$, counting multiplicity.
Using the notation $\log^{+}(x) = \max(0, \log x)$,
(where $x > 0$ and log is the real logarithm function),
we put for $r \in]0, R[$,

$$m(r,f) = \log^{+} |f|(r),$$

$$T(r,f) = N(r,f) + m(r,f).$$

1 -Introduction and preliminary results

The function $r \to T(r, f)$ is called the Nevanlinna function (also called the characteristic function of f).
With the above notations, using the fact that

$$\log x = \log^+ x - \log^+ \frac{1}{x}, \text{ for } x > 0,$$

we can rewrite Proposition (1.1) as follows :

proposition 1.2

Let $f \in \mathcal{M}(D^-(0, R))$ be such that 0 is neither a zero nor a pole of f. Then, for every $r \in]0, R[$, we have

$$T(r, \frac{1}{f}) = T(r, f) + O(1).$$

we have also The following properties.

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proposiotion 1.3

Let $f_1, ..., f_k \in \mathcal{M}(D^-(0,R))$. We assume that the functions f_i, $(1 \leq i \leq k)$, $f_1 + ... + f_k$, and $f_1 ... f_k$ have no zero and no pole at the origin. Then, for every $r \in]0, R[$, we have

$$m(r, f_1 + ... + f_k) \leq m(r, f_1) + ... + m(r, f_k),$$

$$m(r, f_1 ... f_k) \leq m(r, f_1) + ... + m(r, f_k),$$

and

$$N(r, f_1 + ... + f_k) \leq N(r, f_1) + ... + N(r, f_k),$$

$$N(r, f_1 ... f_k) \leq N(r, f_1) + ... + N(r, f_k),$$

and

$$T(r, f_1 + ... + f_k) \leq T(r, f_1) + ... + T(r, f_k),$$

$$T(r, f_1 ... f_k) \leq T(r, f_1) + ... + T(r, f_k).$$

1 -Introduction and preliminary results

proposition 1.4

Let $f \in \mathcal{M}(D^{-}(0,R))$ and $a \in K$ such that 0 neither a zero nor a pole of f.
Then for every $r \in]0,R[$, we have :
(*i*) $\quad T(r,af) = T(r,f) + O(1)$,
(*ii*) $\quad T(r,f-a) = T(r,f) + O(1)$.

The following property was proved in [5].

proposition 1.5

Let $f \in \mathcal{M}(K)$ such that $f(0) \neq 0$ and $f(0) \neq \infty$, we have the following equivalences
(*i*) $\quad f$ is a constant $\Leftrightarrow T(r,f) = o(\log r), \ r \to +\infty$,
(*ii*) $\quad f$ belongs to $K(x) \ \Leftrightarrow T(r,f) = O(\log r), \ r \to +\infty$,
(*iii*) $\quad f$ is non-constant $\Leftrightarrow$ there is $C \in \mathbb{R}$ and $A > 0$ such that

$$T(r,f) \geq \log r + C, \ \text{pour } r > A.$$

1 -Introduction and preliminary results

Let us, finally, recall that the order $\rho(\varphi)$ of meromorphic function $\varphi \in \mathcal{M}(\ K)$ is defined by,

$$\rho(\varphi) = \overline{\lim}_{r \to +\infty} \frac{\log T(r, \varphi)}{\log r}.$$

2 -$q-$difference equations

Let us now return to our main problem. In [3], N. Boudjrida, A. Boutabaa and S. Medjerab studied the equation :

$$(E) \qquad \sum_{i=0}^{s} A_i(x)f(q^i x) = B(x),$$

where $q \in K$ is such that $0 < |q| < 1$
and $A_0(x), \cdots, A_s(x)$, $B(x)$ are elements of $K(x)$.
They obtained the following result :

theorem A

If $f \in \mathcal{M}(K)$ is a solution of Equation (E), then :
$T(r, f) = O((\log r)^2)$, $r \to +\infty$.

2 -$q-$difference equations

We observe that Equation (E) considered above has coefficients $B, A_0, ..., A_s$ satisfying $T(r, A_i) = O(\log r)$, $T(r, B) = O(\log r)$ and that the solutions f must be $O((\log r)^2)$ when $r \to +\infty$.
This raises a question about the order of growth of meromorphic solutions f of Equation (E) if $A_0(x)$, ..., $A_s(x), B(x)$ are replaced by more general meromorphic functions.
In this work, we aim to generalize Theorem A, in the following way.
Let us consider the q-difference equation :

$$\sum_{i=0}^{s} g_i(x)y(q^i x) = h(x) \qquad (2.1)$$

where $q \in K$ is such that $0 < |q| < 1$ and $h(x)$, $g_0(x)$, ..., $g_s(x)$ $(s \geq 1)$ are elements of $\mathcal{M}(K)$ such that $g_0(x)g_s(x) \not\equiv 0$.

2 -$q-$difference equations

Let us denote by $T(r)$ the function,

$$T(r) = \max\{T(r,h);\ T(r,g_0);\cdots;\ T(r,g_s)\},\ \text{for every } r > 0.$$

Then we have :

theorem 2.1

If $f \in \mathcal{M}(K)$ is a solution of Equation (2.1), we have :

$$T(r,f) = O(T(r)\log r),\ \text{when } r \to +\infty.$$

2 -$q-$difference equations

As an immediate consequence of Theorem 2.1, we have the following result :

corollary 2.1

Let $\kappa = \max\{\rho(h);\ \rho(g_0);\ \rho(g_s)\}$ be the maximum of the orders of the functions $h, g_0, \cdots, g_s$ of Equation (2.1).
Then the order $\rho(f)$ of every meromorphic solution f of Equation (2.1) satisfies $\rho(f) \leq \kappa$.

2 -$q-$difference equations

More particularly, we have the following result which is yet an important generalization of Theorem A :

corollary 2.2

If the functions $h, g_0, \cdots, g_s$ above are of zero-order, then every meromorphic solution f of Equation 2.1 is of zero-order.

remark 2.1

Using Nevanlinna Theory in C, the same method of proof enables us to show that Theorem 2.1 and its corollaries are true in the complex case.

3 -Difference equations

Let us now consider the difference equation

$$\sum_{i=0}^{s} g_i(x)y(x+i) = h(x) \qquad (3.2)$$

where $h(x)$, $g_0(x)$, ..., $g_s(x)$ $(s \geq 1)$ are elements of $\mathcal{M}(\ K)$ such that $g_0(x)g_s(x) \not\equiv 0$.

Let $T(r) := \max\{T(r, g_0), \cdots, T(r, g_s), T(r, h)\}$

and let $M(r) := \max\limits_{0 \leq i \leq s} \{|g_i|(r)\}$.

In what follows, the goal is to study the growth of meromorphic functions $y = f(x)$ that are solutions of Equation (3.2) according to that of the functions $g_0, ..., g_s, h$.

3 -Difference equations

theorem 3.1

Suppose that each of the functions $g_0, ..., g_s, h$ in Equation (3.2) admits finitely many poles. Suppose that there exists a constant $C > 0$ such that, for $r > 0$ big enough,
we have

$$|\sum_{i=0}^{s} g_i|(r) \geq CM(r).$$

Then every meromorphic solution f of Equation (3.2) is such that

$$T(r, f) \leq T(r, h) + \min_{0 \leq i \leq s} T(r, g_i) + O(\log r) \leq 2T(r) + O(\log r)$$

$$r \to +\infty.$$

The following result matches to a particular situation where conditions of Theorem 3.1 are fulfilled.

3 -Difference equations

corollary 3.1

Let the functions $g_0, ..., g_s, h$ in Equation (3.2) be meromorphic with finitely many poles. Suppose that there exists an integer ℓ, $0 \leq \ell \leq s$, such that $|g_\ell|(r) > \max_{0\leq i\leq s,\, i\neq\ell}\{|g_i|(r)\}$.

Then every meromorphic solution f of Equation(3.2) is such that :

$$r \to +\infty.$$

$$T(r,f) \leq T(r,h) + \min_{0\leq i\leq s} T(r,g_i) + O(\log r) \leq 2T(r) + O(\log r),$$

As an immediate consequence, we have :

corollary 3.2

Under the assumption of Theorem 3.1,
let $\kappa = \max\{\rho(h),\ \rho(g_0), ...,\ \rho(g_s)\}$. Then every meromorphic solution f of Equation (3.2) satisfies $\rho(f) \leq \kappa$.

3 -Difference equations

More particularly, we have :

corollary 3.3

If there exists an integer ℓ, $0 \leq \ell \leq s$, such that $|g_\ell|(r) > \max_{0 \leq i \leq s,\, i \neq \ell} \{|g_i|(r)\}$, then for every meromorphic solution f of Equation (3.2) we have $\rho(f) \leq \max\{\rho(h), \rho(g_\ell)\}$.

remark 3.1

The results above are false in the complex case. Indeed, for instance, Yik-Man Chiang and Shao-Ji Feng ([8]) proved that, if $h, g_0, ..., g_s \in \mathcal{A}(\ C)$ and if there exists a unique integer ℓ, $0 \leq \ell \leq s$, such that $\rho(g_\ell) = \kappa = \max_{0 \leq i \leq s} \{\rho(g_i)\}$,
then, for every meromorphic solution (in C) to Equation (3.2), we have $\rho(f) \geq \kappa + 1$.

3 -Difference equations

To state some results related to the situation where the coefficients of Equation (3.2) are rational functions, we first give a definition.

definition 3.1

If $P(x); Q(x)$ are polynomials of $K[x]$ without common zeros, we call degree of the rational function $R(x) = \frac{P(x)}{Q(x)}$ and denote by $\deg R(x)$ the number

$$\deg R(x) = \max\{\deg P(x), \deg Q(x)\}$$

.

3 -Difference equations

We have :

corollary 3.4

Let us consider the equation :

$$\sum_{i=0}^{s} R_i(x)y(x+i) = R(x) \qquad (3.3)$$

where $R(x), R_0(x), ..., R_s(x)$ are rational functions over K such that $R_0(x)R_s(x) \not\equiv 0$. If $\deg\left(\sum_{i=0}^{s} R_i\right) = \max_{0\leq i\leq s}\{\deg R_i\}$, then every meromorphic solution f of this equation is a rational function such that : $\deg f(x) \leq \max_{0\leq i\leq s}\{\deg R_i(x), R(x)\}$.

3 -Difference equations

proof

Indeed, as $R(x), R_0(x), ..., R_s(x)$ are rational functions, there exists $\gamma > 0$, $\gamma_i > 0$ $(i = 0, ...s)$ such that, for $r > 0$,
$|\sum_{i=0}^{s} R_i|(r) = \gamma r^{\deg(\sum_{i=0}^{s} R_i)}$ and $|R_i|(r) = \gamma_i r^{\deg R_i}$.
Hence

$$\max_{0\leq i\leq s} |R_i|(r) = \max_{0\leq i\leq s}\{\gamma_i r^{\deg R_i}\} \leq \lambda\, r^{\max\{\deg |R_i|\}} = \lambda\, r^{\deg(\sum_{i=0}^{s} R_i)}$$

$$= \frac{\lambda}{\gamma}|\sum_{i=0}^{s} R_i|(r).$$

It follows that
$|\sum_{i=0}^{s} R_i|(r) \geq C \max_{0\leq i\leq s} |R_i|(r)$, with $C = \frac{\gamma}{\lambda} > 0$.

3 -Difference equations

Using Theorem 3.1, we obtain that, if $f \in \mathcal{M}(K)$ is a solution of Equation (3.3) hence
$T(r,f) \leq 2\max\{T(r,R), \min_{0\leq i\leq s} T(r,R_i)\} + O(\log r) = O(\log r),\;$ when $r \to +\infty$.
Thus $f \in\; K(x)$.
On the other hand, for r big enough we have
$T(r,f) \leq 2\max_{0\leq i\leq s}\{T(r,R_i), T(r,R)\} + O(\log r)$.
Thus $(\deg f)\log r + O(1) \leq \max_{0\leq i\leq s}\{(\deg R_i)\log r + O(1), (\deg R)\log r + O(1)\}$
It follows that, for $r > 0$ big enough
$\deg f(x) \leq \max_{0\leq i\leq s}\{\deg R_i(x), \deg R(x)\}$.

Corollary 3.4 is false in $\;C$.

3 -Difference equations

Indeed we have the following Example

example 3.1

In C, the difference equation

$$f(x) - \frac{x+1}{xe} f(x+1) = \frac{1}{x}\left(\frac{e-1}{e}\right),$$

satisfies all the conditions above but has a transcendental meromorphic solution $g(x) = \frac{e^x+1}{x}$.

3 -Difference equations

corollary 3.5

Let L be an algebraically closed field of characteristic zero. Let the equation

$$\sum_{i=0}^{s} R_i(x)y(x+i) = R(x) \qquad (3.4)$$

,

where $R(x), R_0(x), ..., R_s(x)$ are rational functions over L such that $R_0(x)R_s(x) \not\equiv 0$. If $\deg\left(\sum_{i=0}^{s} R_i\right) = \max_{0\leq i\leq s}\{\deg R_i\}$, then every solution $f \in L(x)$ of Equation (3.4) satisfies

$$\deg f(x) \leq \max_{0\leq i\leq s}\{\deg R_i(x), \deg R(x)\}.$$

3 -Difference equations

proof

Indeed, L equipped with the trivial absolute value defined by

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0, \end{cases}$$

is a complete ultrametric algebraically closed field. We have

$$\mathcal{A}(L) = L[x] \text{ and } \mathcal{M}(L) = L(x).$$

We then apply Corollary 3.4 and obtain that :

$$\deg f(x) \leq \max_{0 \leq i \leq s} \{\deg R_i(x), R(x)\}.$$

Proof of Theorem 2.1

remark4.1

A first observation is that in Equations (2.1), we may assume without loss of generality that $h(x) = 0$ and that the g_i's are entire functions.

Indeed, suppose that $h(x) \neq 0$ and that $f(x)$ is a meromorphic solution of Equation (2.1). Then, it is easily seen that $f(x)$ is a solution of the nontrivial equation :

$$h(x)\sum_{i=0}^{s} g_i(qx)f(q^{i+1}x) - h(qx)\sum_{i=0}^{s} g_i(x)f(q^{i}x) = 0.$$

Proof of Theorem 2.1

We may also assume that the g_i's are entire functions. So, we reduce our study to the following equation :

$$\sum_{i=0}^{s} g_i(x)y(q^i x) = 0 \qquad (4.5)$$

,

where $q \in K$ is such that $0 < |q| < 1$ and $g_0(x)$, ..., $g_s(x)$ $(s \geq 1)$ are elements of $\mathcal{A}(K)$ such that $g_0(x)g_s(x) \not\equiv 0$

Proof of Theorem 2.1

In what follows, we denote by σ_q the function defined for all $x \in K$ by $\sigma_q(x) = qx$. Then, for every $k \in N^*$, $\sigma_q^k = \sigma_q \circ \cdots \circ \sigma_q$ is obtained by applying k times the function σ_q . We agree that $\sigma_q^0 = Id$ where Id is the identity function in K.

Some properties of these operators are summarized in the following Lemma whose proof is easily checked :

lemma 4.1

For every $f \in \mathcal{M}(K)$, every $r > 0$ and every $n \in N$, we have :

(1) $|f \circ \sigma_q^n|(r) = |f|(|q|^n r)$,

(2) $m(r, f \circ \sigma_q^n) = m(|q|^n r, f)$,

(3) $N(r, f \circ \sigma_q^n) = N(|q|^n r, f)$,

(4) $T(r, f \circ \sigma_q^n) = T(|q|^n r, f)$.

Proof of Theorem 2.1

The following result will be useful for the proof of Theorem 2.1 :

proposition 4.1

Let f be a meromorphic solution in K of Equation (2.1).
Then, if α is a non-zero pole of f, there exists an integer $m \in N$ and a zero θ of g_0 different from zero such that $\alpha = q^{-m}\theta$ and $\omega_\theta(g_0) + \omega_\alpha(f) \geq 0$.

Proof of Theorem 2.1

proof

If $\omega_\alpha(g_0) + \omega_\alpha(f) \geq 0$, we are done because it suffices to take $\theta = \alpha$. Suppose that $\omega_\alpha(g_0) + \omega_\alpha(f) < 0$. This means that α is a pole of $g_0(x)f(x)$, then there exists at least an index $i_1 \in \{1, ..., s\}$ such that $g_{i_1}(\alpha)f(q^{i_1}\alpha) = \infty$, and particularly $\alpha_1 = q^{i_1}\alpha$ is a pole of f. If $\omega_{\alpha_1}(g_0) + \omega_{\alpha_1}(f) < 0$, we find in the same way an index $i_2 \in \{1, ..., s\}$ such that $\alpha_2 = q^{i_2}\alpha_1 = q^{i_1+i_2}\alpha$ is a pole of f, etc... As we cannot have a sequence of poles of f with strictly decreasing moduli, the above process must stop at a certain rank, and this completes the proof of our assertion.

Proof of Theorem 2.1

As indicated in Remark 3.1, the whole problem is reduced to the case of Equation (4.5).
Hence, let $f \in \mathcal{M}(K)$ be a solution of Equation (E). We may also suppose that f has no pole at the origin.

Let us first estimate $N(f, r)$.

If $g_0(x)$ has no zero different from 0 we see, by Proposition 4.1, that the function $f \in \mathcal{M}(K)$ is entire and hence $N(f, r) = 0$.
Suppose then that $g_0(x)$ admits at least one zero different from 0 and let $\rho = \min\{|x| \ / \ x \in K \setminus \{0\}$ and $g_0(x) = 0\}$.

Proof of Theorem 2.1

For every $r > 0$, we have :

$$N(r, f) = -\sum_{0<|\alpha|\leq r, f(\alpha)=\infty} \omega_\alpha(f) \log \frac{r}{|\alpha|}.$$

But, by Proposition 4.1, every pole α of f in $D(0, r) \setminus \{0\}$ is of the form $\alpha = q^{-n}\beta$ with some $n \in N$ and some β in $D(0, r) \setminus \{0\}$ such that $g_0(\beta) = 0$. This implies that : $0 \leq n \leq \left[\frac{1}{\log |q|} \log \frac{\rho}{r}\right]$, (where $[t]$ denotes the integral part of the real number t).

Proof of Theorem 2.1

Hence :

$$N(r,f) \leq \left(\left[\frac{1}{\log|q|}\log\frac{\rho}{r}\right]+1\right)\sum_{0<|\beta|\leq r, g_0(\beta)=0}\omega_\beta(g_0)\log\frac{r}{|\beta|},$$

i.e,

$$N(r,f) \leq \left(\left[\frac{1}{\log|q|}\log\frac{\rho}{r}\right]+1\right)N(\frac{1}{g_0},r).$$

By hypothesis, we have

$N(\frac{1}{g_0},r) \leq T(r,g_0)+O(1) \leq T(r)+O(1), r \to +\infty.$

We also see that

$$\left[\frac{1}{\log|q|}\log\frac{\rho}{r}\right]+1 = O(\log r),\ \ r \to +\infty.$$

It follows that :

(1) $$N(r,f) = O(T(r)\log r),\ \ r \to +\infty.$$

Proof of Theorem 2.1

Let us now estimate $\log|f|(r)$.

Without loss of generality, we may suppose that f has no zero and no pole at the origin. Hence there exists $\epsilon > 0$ such that f has no zeros and no poles in $D(0,\epsilon)$, so that $|f|(t)$ is constant for $0 \leq t \leq \epsilon$.

From Equation (E), we have for every $r > 0$:

$$|f|(r) \leq \max\left\{ |\frac{g_1}{g_0}|(r)|f|(|q|r),\ |\frac{g_2}{g_0}|(r)|f|(|q|^2 r),\ \cdots\ |\frac{g_s}{g_0}|(r)|f|(|q|^s r) \right\}.$$

As $g_0, \cdots, g_s$ lie in $\mathcal{A}(\ K)$ and satisfy $T(r, g_i) \leq T(r), \forall i$, we have for $r > 0$ big enough :

(2) $$|f|(r) \leq e^{T(r)} \max\left\{ |f|(|q|r),\ |f|(|q|^2 r),\ \cdots,\ |f|(|q|^s r) \right\}.$$

Proof of Theorem 2.1

Let us take r big enough in order to assure that the integer $k = \left[\frac{\log r - \log \epsilon}{-\log |q|}\right] + 1$ is $\geq s$, we have by (2) :

(3) $$|f|(r) \leq e^{T(r)} \max\Big\{ |f|(|q|r),\ |f|(|q|^2 r), \cdots, \||f|(|q|^s r),\ \cdots,\ |f|(|q|^k r) \Big\}.$$

Proof of Theorem 2.1

Let us set :

$$\begin{cases} \mu_1 &= |f|(|q|^k r), \\ \mu_2 &= \max\{|f|(|q|^{k-1} r),\ |f|(|q|^k r)\}, \\ \vdots & \\ \mu_{k-1} &= \max\{|f|(|q|^2 r), \cdots, |f|(|q|^s r), \cdots, |f|(|q|^k r)\}, \\ \mu_k &= \max\{|f|(|q| r),\ |f|(|q|^2 r), \cdots, |f|(|q|^s r), \cdots, |f|(|q|^k r)\}. \end{cases} \quad \textbf{(4)}$$

Hence, (3) becomes :

$$\textbf{(5)} \qquad |f|(r) \leq e^{T(r)} \mu_k.$$

Proof of Theorem 2.1

On the other hand, we have : $|q|\epsilon \leq |q|^k r < \epsilon$. Hence, using the fact that $|f|(t)$ is constant for $0 \leq t \leq \epsilon$, we have :

(6)
$$|f|(|q|^k r) = |f|(|q|^{k+1} r) = |f|(|q|^{k+2} r) = \cdots = \mu_1 = C = \text{Constant}.$$

Then replacing r in (3) successively by $|q|r$, $|q|^2 r$, $\cdots$, $|q|^{k-1}r$ we obtain :

(7)
$$\begin{cases} |f|(|q|r) \leq e^{T(|q|r)}\mu_{k-1}, \\ |f|(|q|^2 r) \leq e^{T(|q|^2 r)}\mu_{k-2}, \\ \vdots \\ |f|(|q|^{k-2} r) \leq e^{T(|q|^{k-2} r)}\mu_2, \\ |f|(|q|^{k-1} r) \leq e^{T(|q|^{k-1} r)}\mu_1. \end{cases}$$

Proof of Theorem 2.1

It follows by (4) and (7) that for $r > 0$ big enough :

(8)

$$\begin{cases} \mu_1 & = C, \\ \mu_2 & \leq e^{T(|q|^{k-1}r)}\mu_1 \\ \vdots & \\ \mu_{k-1} & \leq e^{T(|q|^2 r)}\mu_{k-2}, \\ \mu_k & \leq e^{T(|q|r)}\mu_{k-1}. \end{cases}$$

From (5) and (8) we have : $|f|(r) \leq e^{\sum_{i=0}^{k-1} T(|q|^i r)} C.$
As $T(|q|^i r) \leq T(r), \forall i = 0, \cdots k-1$, it follows that :

(9) $$|f|(r) \leq e^{kT(r)} C.$$

As $k = O(\log r)$, when $r \to +\infty$, it easily follows from this that :

(10) $$\log^+ |f|(r) = O(T(r) \log r), \ r \to +\infty.$$

Proof of Theorem 2.1

Finally, relations (1) and (10) yield :

(11) $$T(r,f) = O(T(r)\log r),\ \ r \to +\infty.$$

This completes the proof of Theorem 2.1

5 -Proof of Theorem 3.1

First we state some lemmas whose proof is easily checked.

lemma 5.1

Let $f \in \mathcal{M}(K)$ and let $\alpha, \zeta \in K$. Then ζ is a pole of f of order ℓ if, and only if, $\zeta - \alpha$ is a pole of $f(x+\alpha)$ of order ℓ.

In what follows, we denote by τ the function defined for all $x \in K$ by $\tau(x) = x + 1$. For every $k \in N^*$, $\tau^k = \tau \circ \cdots \circ \tau$ is obtained by applying k times the function τ . We agree that $\tau^0 = Id$ where Id is the identity function in K.

Proof of Theorem 3.1

some properties of these operators are summarized in the following Lemma :

Lemma 5.2

For every $f \in \mathcal{M}(\ K)$ and every $r > 1$ and every $k \in \ N$ we have :

(1) $|f \circ \tau^k|(r) = |f|(r)$,

(2) $m(r, f \circ \tau^k) = m(r, f)$,

(3) $N(r, f \circ \tau^k) = N(r, f) + O(1)$,

(4) $T(r, f \circ \tau^k) = T(r, f) + O(1)$,

We have also :

Lemma 5.3

Let $f \in \mathcal{M}(\ K)$ and let $\Delta f = f \circ \tau - f$. For $r > 0$, we have

$$|\Delta(f)|(r) \leq \frac{|f|(r)}{r}.$$

Proof of Theorem 3.1

proof

Let $f \in \mathcal{A}(\ K)$ be defined by $f(x) = \sum_{n\geq 0} a_n x^n$. Hence, we have :

$\Delta(f)(x) = \sum_{n\geq 1} a_n \sum_{k=0}^{n-1} x^k (x+1)^{n-1-k}$.

Thus, for $r > 1$, we have : $|\Delta(f)|(r) \leq \max_{n\geq 1}\{|a_n| r^{n-1}\} \leq \frac{|f|(r)}{r}$.

Let now $f \in \mathcal{M}(\ K)$ be such that $f(x) = \frac{\zeta(x)}{\xi(x)}$, where $\zeta, \xi \in \mathcal{A}(\ K)$.

We have :

$\Delta(f)(x) = \frac{\Delta(\zeta)(x)\xi(x) - \zeta(x)\Delta(\xi)(x)}{\xi(x)\xi(x+1)}$.

It follow that $|\Delta(f)|(r) = \frac{|\Delta(\zeta)\xi - \zeta\Delta(\xi)|(r)}{|\xi|(r)|\xi(x+1)|(r)} \leq \frac{|\zeta|(r)|\xi|(r)}{r|\xi|(r)^2} = \frac{|f|(r)}{r}$.

The proof of Theorem 3.1 is based upon the following propositions.

Proof of Theorem 3.1

proposition 5.1

Let the functions $g_0, ..., g_s, h$ in Equation (3.2) be meromorphic with finitely many poles. Then, if f is a mermorphic solution of this equation, we have

$$N(r, f) \leq \min_{0 \leq i \leq s} Z(r, g_i) + O(\log r), \ \ r \to +\infty.$$

Proof of Theorem 3.1

proof

Let j be a fixed integer among $0, \cdots, s$. Let α be a pole of f such that $|\alpha| > R$. Then $\beta := \alpha - j$ is a pole of $f \circ \tau^j$ such that $\omega_\alpha(f) = \omega_\beta(f \circ \tau^j)$.
We have :

(5.1) $\qquad -\omega_\alpha(f) \leq \omega_\beta(g_j)$.

Indeed, suppose that $-\omega_\alpha(f) > \omega_\beta(g_j)$. It follows that $\omega_\beta(g_j) + \omega_\beta(f \circ \tau^j) < 0$. Since we have, from Equation (3.2) : $g_j(x)(f \circ \tau^j)(x) = h(x) - \sum_{i \neq j} g_i(x)(f \circ \tau^i)(x)$, it follows that there exists an integer $k \in \{0, \cdots, s\} \setminus \{j\}$ such that $\omega_\beta(g_k) + \omega_\beta(f \circ \tau^k) \leq \omega_\beta(g_j) + \omega_\beta(f \circ \tau^j)$. As $k \neq j$, we have $\omega_\beta(f \circ \tau^k) \geq 0$ and hence $\omega_\beta(g_k) \leq \omega_\beta(g_j) + \omega_\beta(f \circ \tau^j) < 0$, a contradiction because β is not a pole of g_k.

Proof of Theorem 3.1

From (5.1), we deduce that :

(5.2) $N(r, f) \leq Z(r, g_j) + O(\log r), \;\; r \to +\infty$ for every $j = 0, \cdots, s$, and finally :

(5.3) $N(r, f) \leq \min\limits_{0 \leq j \leq s} Z(r, g_j) + O(\log r), \;\; r \to +\infty.$

This completes the proof of the Proposition.

(Remark 5.1)

It follows from the Proposition above that, if one of the functions g_i, $(i = 0, \cdots, s)$, is a rational function, then the function f has finitely many poles.

Proof of Theorem 3.1

proposition 5.2

Let the functions $g_0, ..., g_s, h$ in Equation (3.2) be meromorphic. Suppose that there exists a constant $C > 0$ such that

$$|\sum_{i=0}^{s} g_i|(r) \geq CM(r).$$

Then every meromorphic solution f of this equation satisfies

$$|f|(r) \leq \frac{|h|(r)}{CM(r)}.$$

Proof of Theorem 3.1

proof

Let $f \in \mathcal{M}(\ K)$ be a solution of Equation (3.2). Then we have :

(5.4) $\quad g_0(x) + \sum_{i=1}^{s} g_i(x)\frac{f(x+i)}{f(x)} = \frac{h(x)}{f(x)}.$

Using Lemma 5.3, we show that $\lim\limits_{|x|\to+\infty} \frac{f(x+1)}{f(x)} = 1.$

It follows that, for every $i \in \{0, \cdots, s\}$, we have $\lim\limits_{|x|\to+\infty} \frac{f(x+i)}{f(x)} = 1.$

Hence by (5.4), we have $g_0(x) + \sum_{i=1}^{s} g_i(x)(\varepsilon_i(x) + 1) = \frac{h(x)}{f(x)}$, where $\varepsilon_i(x)$ are meromorphic functions such that $\lim_{|x|\to+\infty} \varepsilon_i(x) = 0$,

i.e, $\quad \sum\limits_{i=0}^{s} g_i(x) + \sum\limits_{i=1}^{s} g_i(x)\varepsilon_i(x) = \frac{h(x)}{f(x)}.$

Proof of Theorem 3.1

For $r > 0$ big enough and for $i = 1, ..., s$, we have : $|\varepsilon_i|(r) < C$.
It follows that

$$|\sum_{i=1}^{s} g_i(x)\varepsilon_i(x)|(r) < C \max_{1\leq i\leq s} |g_i|(r) \leq C \max_{0\leq i\leq s} |g_i|(r) = CM(r).$$

As, by hypothesis, $|\sum_{i=0}^{s} g_i|(r) \geq CM(r)$, we deduce that

$$|\frac{h}{f}|(r) = |\sum_{i=0}^{s} g_i|(r) \geq CM(r).$$

We are now able to prove the main result of this part.

Proof of Theorem 3.1

proof of theorem 3.1 : By Proposition 5.1, we have :
$N(r,f) \leq \min_{0\leq i\leq s} Z(r,g_i) + O(\log r)$, $r \to +\infty$, i.e.,

(1) $N(r,f) \leq \min_{0\leq i\leq s} N(r,\frac{1}{g_i}) + O(\log r)$, $r \to +\infty$..

On the other hand, by Proposition 5.2, for $r > 0$ we have $|f|(r) \leq \frac{|h|(r)}{CM(r)}$. It follows that :

$$\begin{aligned} \log|f|(r) &\leq \log|h|(r) - \max_{0\leq i\leq s}\{\log|g_i|(r)\} - \log C, \\ &= \log|h|(r) + \min_{0\leq i\leq s}\{\log\frac{1}{|g_i|(r)}\} - \log C. \end{aligned}$$

Hence

(2)
$\log^+|f|(r) \leq \log^+|h|(r) + \min_{0\leq i\leq s}\{\log^+\frac{1}{|g_i|(r)}\} + O(1)$, $r \to +\infty$.

Proof of Theorem 3.1

Finally, by relations (1) and (2), we obtain that, for $r \to +\infty$,

$$T(r,f) = N(r,f) + \log^+ |f|(r)$$

$$\begin{aligned}
&\leq \log^+ |h|(r) + \min_{0 \leq i \leq s} (N(r, \frac{1}{g_i}) + \log^+ \frac{1}{|g_i|(r)}) + O(\log r),\\
&\leq \log^+ |h|(r) + \min_{0 \leq i \leq s} \{T(r, \frac{1}{g_i})\} + O(\log r),\\
&\leq \log^+ |h|(r) + \min_{0 \leq i \leq s} \{T(r, g_i)\} + O(\log r),
\end{aligned}$$

Thus Theorem 3.1 is proved

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★★★ *Thank you* ★★★