p-adic Quantum Mechanics and Quantum Channels

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QM & p-Adic QM. Standard statistical model.

Let $\mathcal{H}$ be a **separable complex Hilbert space**.
State ρ of the QM system $\equiv$ density operator in $\mathcal{H}$, $\rho \in \mathfrak{S}(\mathcal{H})$.
Let (X, Σ) be a **measurable space**.
Observable $\equiv$ projector-valued measure E on (X, Σ).
The probability distribution of the observable E in the state ρ is defined by the Born-von Neumann formula

$$\mu_\rho^E(B) = \mathrm{Tr}\rho E(B), B \in \Sigma.$$

$(X, \Sigma) = (\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv$ standard statistical model of QM.
$(X, \Sigma) = (\mathbb{Q}_p, \mathcal{B}(\mathbb{Q}_p)) \equiv$ p-adic statistical model of QM.
$\mathbb{R}$ and $\mathbb{Q}_p$ are Borel-isomorphic.

Example of the observable «inspired by p-adics».

- $\mathcal{H} = L^2(\mathbb{Q}_p)$
- $(X, \Sigma) = (\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p))$
- $E(B)f(x) = h_B(x)f(x), B \in \mathcal{B}(\mathbb{Z}_p), x \in \mathbb{Z}_p, f \in \mathcal{H}$

Let $F \colon \mathbb{Z}_p \to \mathbb{R}$ be bounded measurable function.

$$M_F = \int_{\mathbb{Z}_p} F(\lambda)dE(\lambda), M_F f(x) = F(x)f(x), f \in \mathcal{H}.$$

M_F is the bounded selfadjoint operator.
Let A denotes the C^*-algebra generated by operators $E(B), B \in \mathcal{B}(\mathbb{Z}_p)$

$$A \simeq C(\mathbb{Z}_p) \simeq C(\text{Cantor-like subset of } \mathbb{R}).$$

Spectrum of M_F is the Cantor-like subset of $\mathbb{R}$ («p-adic specrtum» of M_f is $\mathbb{Z}_p$).

Quantum channels

Let $\mathcal{H}$ be a complex Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of bounded operators in $\mathcal{H}$ and $\mathfrak{T}(\mathcal{H})$ the ideal of trace-class operators.
Channel $\Phi \equiv$ linear completely positive and trace-preserving map $\Phi\colon \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$.
«Completely positive» means that $\Phi \otimes \mathrm{Id}_d$ is positive for all $d = 1, 2, \dots$.

Quantum channels

- Unitary channel
  $\Phi[\rho] = U\rho U^{-1}$
- von Neumann measurement
  $\Phi[\rho] = \sum_j E_j \rho E_j, \{E_j\}$ – orthogonal resolution of the identity
- Entanglement-breaking channel
  $\Phi[\rho] = \sum_j S_j \operatorname{Tr} \rho M_j, \{M_j\}$ – resolution of the identity
- Kraus decomposition
  $\Phi[\rho] = \sum_j V \rho V^*, \sum_j V^* V = 1$

Additivity problem

χ-capacity of Φ (Holevo capacity):

$$C_\chi(\Phi) = \sup_{\{\rho_i, \pi_i\}} \left(H\left(\Phi\left[\sum_i \pi_i \rho_i\right]\right) - \sum_i \pi_i H\left(\Phi\left[\rho_i\right]\right) \right)$$

Here $H(\rho) = -\operatorname{Tr} \rho \log \rho$ and $\{\rho_i, \pi_i\}$ is a finite set of states $\{\rho_1, \dots \rho_n\}$ with probabilities $\{\pi_1, \dots \pi_n\}$.

$$C_\chi\left(\Phi^{\otimes n}\right) \stackrel{?}{=} nC_\chi(\Phi).$$

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p-adic symplectic geometry

Let F be a 2-dimentional linear space over $\mathbb{Q}_p$, Δ be a non-degenerate antisymmetric ($\equiv$ symplectic) form on F.

- Lattice $L \equiv$ 2-dimentional $\mathbb{Z}_p$ submodule of F, $L = p^m\mathbb{Z}_p \bigoplus p^n\mathbb{Z}_p$.
- Dual lattice $L^* \equiv \{z \in F, \Delta(z,u) \in \mathbb{Z}_p \forall u \in L\}$, $L^* = p^{-n}\mathbb{Z}_p \bigoplus p^{-m}\mathbb{Z}_p$.
- Selfdual lattice $L = L^*$
- Volume of L $|L| = p^{-m-n}$, $L = L^*$ iff $|L| = 1$.
- Symplectic group $Sp(F) \equiv SL_2(\mathbb{Q}_p)$, $|gL| = |L|, g \in Sp(F)$.

Weyl system $\equiv$ Representation of CCR.

Definition

The pair $(W, \mathcal{H})$ is said to be the Weyl system if

- $W : F \to \mathfrak{B}(\mathcal{H})$
- $W(-z) = W^*(z), z \in F$
- $W(z)W(z') = \chi(\Delta(z, z'))W(z')W(z), z, z' \in F$
- $\forall \phi, \psi \in \mathcal{H}$ the function $< \phi, W(z)\psi >: F \to \mathbb{C}$ is measurable

Here $\chi(x) = \exp(2\pi i\{x\}_p)$, $x \in \mathbb{Q}_p$.

The Bohner-Khinchin theorem I.

Function $f : F \to \mathbb{C}$ is positive definite if $\forall z_1, \ldots, z_n \in F$ and $\forall c_1, \ldots, c_n \in \mathbb{C}$

$$\sum_i c_i c_j^* f(z_i - z_j) \geq 0.$$

Function $f : F \to \mathbb{C}$ is Δ-positive definite if $\forall z_1, \ldots, z_n \in F$ and $\forall c_1, \ldots, c_n \in \mathbb{C}$

$$\sum_i c_i c_j^* f(z_i - z_j) \chi \left(\frac{1}{2} \Delta(z_i, z_j) \right) \geq 0.$$

Let ρ be a state in $\mathcal{H}$, W be an irreducible representation of CCR. ρ is uniquely defined by its characteristic function

$$\pi_\rho(z) = \mathrm{Tr}(\rho W(z)).$$

The Bohner-Khinchin theorem II.

Theorem

$\pi(z)$ is characteristic function of a quantum state iff

- $\pi(0) = 1$, $\pi(z)$ is continuous at $z = 0$,
- $\pi(z)$ is Δ-positive definite.

Theorem

Let L be a selfdual lattice F. Then $\forall$ positive definite continuous at $z = 0$ function $\pi(z) : \pi(0) = 1, \operatorname{supp} \pi \subset L$, there exists unique state ρ_π such that

$$\pi(z) = \mathrm{Tr}\left(\rho_\pi W(z)\right).$$

$\forall$ state ρ in $\mathcal{H}$ there exists a unitary operator U in $\mathcal{H}$ such that $\pi_\rho(z) = \mathrm{Tr}\left(U \rho U^{-1} W(z)\right)$ has support in L and is positive definite on L.

p-adic Guassian states I.

Definition

A state ρ is said to be (centered) p-adic Guassian state, if its characteristic function π_ρ will be an indicator function of some lattice L:

$$\pi_\rho = \mathrm{Tr}\left(\rho W(z)\right) = h_L.$$

Let $\mathcal{F}$ be the Fourier transform in $L^2(F)$ defined by the formula

$$\mathcal{F}\left[f\right](z) = \int_F \chi\left(\Delta(z,s)\right) f(s) ds.$$

The following formula is valid

$$|L|^{-1/2}\mathcal{F}\left[h_L\right] = |L^*|^{-1/2} h_{L^*}.$$

We use the notation $\gamma(L)$ for centered Gaussian state defined by lattice L and $\gamma(L,\alpha) = W(\alpha)\gamma(L)W(-\alpha)$ for general Gaussian state.

p-adic Guassian states II.

Theorem

Indicator function h_L of a lattice L defines a state iff $|L| \leq 1$. Gaussian state ρ with characteristic function $\pi_\rho = h_L$ is $|L|P_L$, here P_L is an orthogonal projector of rank $1/|L|$.

Theorem

The following statements are valid.

- *Gaussian state is pure iff the lattice is selfdual.*
- *Entropy of Gaussian state equals $-\log|L|$.*
- *Gaussian states ρ_1 and ρ_2 are unitary equivalent iff $|L_1| = |L_2|$.*
- *Gaussian state has maximun entropy among all states of fixed rank $p^m, m \in \mathbb{Z}_+$.*

p-adic channels

Let $\Phi\colon \rho \to \Phi[\rho]$ be a channel.

- Linear Bosonic channel $\equiv$

$$\pi_{\Phi[\rho]}(z) = \pi_\rho(Kz)k(z),$$

  K – linear transfornation of F, $k\colon F \to \mathbb{C}$.

- Guassian channel $\equiv$ Bosonic channel with $k(z) = h_L(z)$ for some L.

Theorem

Let K be nondegenerate linear transformation of F, L be a lattice in F, $k(z) = h_L(z)$. The formula $\pi_{\Phi[\rho]}(z) = \pi_\rho(Kz)k(z)$ defines a channel iff

$$|L||1 - \det K|_p \leq 1.$$

Additivity of the p-adic Gaussian channels

Theorem

For the p-Adic Gaussian channel the additivity of the χ-capacity holds.

There are two possibilities

- $\Phi[\rho] = \sum_{a\in I} < \phi_a, \rho\phi_a > \gamma(K'L, a)$
  Here $\{\phi_a, a \in I\}$ – orthogonal basis in $\mathcal{H}$, K' – symplectically adjoint to K.
- $\Phi[\rho] = \sum_{\alpha\in J} P^\alpha U\rho U^{-1} P^\alpha$
  $\{P^\alpha, \alpha \in J\}$ – orthogonal resolution of the identity.

p-adic channel with classical noise

p-adic channel with classical noise $\Phi_L \equiv$ linear Bosonic channel with $K = \mathrm{Id}$ and $k(z) = h_L, |L| \leq 1$.

Theorem

Φ_L is an ideal measurement given by the following orthogonal resolution of the identity (instrument)

$$E = \{E_\alpha, \alpha \in F/L^*\},$$

all E_α are of the same dimension $|L|^{-1}$:

$$\Phi_L[\rho] = \sum_{\alpha \in F/L^*} E_\alpha \rho E_\alpha.$$

If $L = L^*$ the measurement is complete.

Entropy gain.

Minimal entropy gain

$$G(\Phi) = \inf_{\rho} \left(H\left(\Phi[\rho]\right) - H(\rho) \right).$$

Theorem

If $\det K \neq 0$ than the following equality holds

$$G(\Phi) = \log|\det K|_p.$$