Generalization of Hensel lemma: finding of roots of p-adic Lipschitz functions

(joint talk with Andrei Khrennikov)

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Outline

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Definitions: p-adic numbers

- For any prime $p \geq 2$ the p-**adic norm** $|\cdot|_p$ is defined in the following way. For every nonzero integer n let $ord_p(n)$ be the highest power of p which divides n, i.e. $n \equiv 0 \pmod{p^{ord_p(n)}}$, $n \not\equiv 0 \pmod{p^{ord_p(n)+1}}$. Then we define $|n|_p = p^{-ord_p(n)}$, $|0|_p = 0$. For rationals $\frac{n}{m} \in \mathbb{Q}$ we set $|\frac{n}{m}|_p = p^{-ord_p(n)+ord_p(m)}$.
- The completion of $\mathbb{Q}$ with respect to the p-adic metric $\rho_p(x,y) = |x-y|_p$ is called the **field of** p-**adic numbers** $\mathbb{Q}_p$. The norm satisfies the strong triangle inequality $|x \pm y|_p \leq \max |x|_p; |y|_p$ where equality holds if $|x|_p \neq |y|_p$.
- The set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}$ is called the **set of** p-**adic integers**.
- Every $x \in \mathbb{Z}_p$ can be expanded in **canonical form**, i.e. in a convergent by p-adic norm series:

$$x = x_0 + px_1 + \ldots + p^k x_k + \ldots, \quad x_k \in \{0, 1, \ldots, p-1\}, k \geq 0.$$

- The **ball** of radius p^{-r} with center a is the set $B_{p^{-r}}(a) = \{x \in \mathbb{Z}_p : |x-a|_p \leq p^{-r}\} = a + p^r \mathbb{Z}_p$.

Definitions: p-adic Lipschitz functions

- Consider functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, which satisfy the Lipschitz condition with constant p^α, $\alpha \geq 0$ (p^α-**Lipschitz functions**).
- Recall that $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is p^α-**Lipschitz function** if

$$|f(x) - f(y)|_p \leq p^\alpha |x - y|_p, \quad \text{for all} \quad x, y \in \mathbb{Z}_p.$$

  This condition is equivalent to that from $x \equiv y \pmod{p^k}$ follows $f(x) \equiv f(y) \pmod{p^{\,k-\alpha}}$ for all $k \geq 1 + \alpha$.
- A class of **1-Lipschitz functions** take a special place (i.e. $\alpha = 0$). For all $k \geq 1$ a 1-Lipschitz transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ the reduced mapping modulo p^k is $f_{modp^k} : \mathbb{Z}/p^k\mathbb{Z} \to Z/p^k\mathbb{Z}$, $z \mapsto f(z) \pmod{p^k}$. Mapping f_{modp^k} is a well-defined (the f_{modp^k} does not depend on the choice of representative in the ball $z + p^k\mathbb{Z}_p$).

Hensel's lifting lemma

The main tool for finding the roots of p-adic functions that map the ring of p-adic integers into itself, is a classical result - **Hensel's lifting lemma**.

Theorem (Hensel's lifting lemma for p-adic case)

Let $f(x) \in \mathbb{Z}_p[x]$ be a polynomial with integer p-adic coefficients and $f'(x) \in \mathbb{Z}_p[x]$ be its formal derivative. Suppose that $\bar{a} \in \mathbb{Z}_p$ is a p-adic integer such that $f(\bar{a}) \equiv 0 \pmod{p}$ and $f'(\bar{a}) \not\equiv 0 \pmod{p}$.
Then there exists a unique p-adic integer $a \in \mathbb{Z}_p$ such that $f(a) = 0$ and $a \equiv \bar{a} \pmod{p}$.

Classical Hensel's lifting lemma

Example. Find the solutions of $x^3 + x^2 + 29 \equiv 0 \pmod{25}$.
Solution. Let $f(x) = x^3 + x^2 + 29$. We see (by inspection) that solutions of $f(x) \equiv 0 \pmod 5$ have $x \equiv 3 \pmod 5$. Because $f^{'}(x) = 3x^2 + 2x$ and $f^{'}(3) = 33 \equiv 3 \not\equiv 0 \pmod 5$, Hensel's lemma tells us that there is a unique solution modulo 25 of the form $3 + 5t$, where $t \equiv -\overline{f'(3)}(\frac{f(3)}{5}) \pmod 5$. Note that $\overline{f'(3)} = \overline{3} = 2$, because 2 is inverse to 3 mod 5. Also note that $\frac{f(3)}{5} = \frac{65}{5} = 13$. It follows that $t \equiv -2 \cdot 13 \equiv 4 \pmod 5$. We conclude that $x \equiv 3 + 5 \cdot 4 = 23$ is the unique solution of $f(x) \equiv 0 \pmod{25}$.

Hensel's lifting lemma

Hensel's lifting lemma (Hll) for p-adic functions characterized by the following circumstances:

1. under restrictions on the function f (value of the derivative f at a given point) Hll gives the **criterion of the existence of the root** of p-adic function f on the set of p-adic integers. By solving a finite number of congruences modulo p we can determine whether f has the root in $\mathbb{Z}_p$;
2. Hll provides an **algorithm for finding the approximate value** (in p-adic metric) of the root of f with a given accuracy. This problem is reduced to solving linear congruences modulo p at each step of the iterative procedure. Solvability of the corresponding linear congruences is determined by the restrictions that are imposed on the derivative f';
3. Hll is **applicable for p-adic functions** defined by **polynomials with integer p-adic coefficients**. Even if the function f is defined by polynomial, but not with the p-adic integer coefficients, the "formal" use of Hensel's lifting lemma to find the approximate value of the root leads to false results.

Question arises:

- How could we find roots of p-adic function that maps p-adic integers into itself and is not defined by a polynomial with integer coefficients, for example, if f is not differentiable at all?
- We solved this problem for the p-adic functions that satisfy the Lipschitz condition with constant p^{α}, $\alpha \geq 0$.

Results: 1-Lipschitz functions

General criterion for the existence of the root of 1-Lipschitz functions:
Theorem 1.
Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be 1-Lipschitz function. The function f has a root if and only if the congruences $f_{modp^k}(x) \equiv 0 \pmod{p^k}$ are solvable for any $k \geq 1$, where $f_{modp^k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}, \;\; f_{modp^k}(z) = f(z) \pmod{p^k}$.

It follows that, in general, for 1-Lipschitz function f by solving a finite number of congruences $\bmod p^k$ it is impossible to determine whether the function f has root in $\mathbb{Z}_p$, as it is done with the use of classical Hensel's lifting lemma.

Roots of 1-Lipschitz functions. Example

Let $p = 5$ and $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ such that

$$f(x) = f(x_0 + 5x_1 + \ldots + 5^k x_k \ldots) = x + 5^N(-x_N + x_N^2 + 3),$$

where N is a fixed positive integer. Note that f is 1-Lipschitz function. We need to solve a finite number of congruences of the form $f_{mod 5^k}(x) \equiv 0 \pmod{5^k}$, $k \leq S$. Let $N > S$. It is clear that $f_{mod 5^k}(0) \equiv 0 \pmod{5^k}$, $k \leq S$. If a direct analogy with the Hensel's lifting lemma is true, then we can assume that $x = 0$ is the root of the function f. However, the congruence $f(x) \equiv 0 \pmod{5^{N+1}}$ has no solutions.

Indeed, suppose that $h \in \{0, \ldots, 5^{N+1} - 1\}$ be a solution of this congruence and $h = \bar{h} + 5^N h_N$. Because $f(h) \equiv f(\bar{h}) \equiv 0 \pmod{5^N}$, then $\bar{h} = 0$ and $f(h) \equiv 5^N(h_N^2 + 3) \equiv 0 \pmod{5^{N+1}}$ or $h_N^2 + 3 \equiv 0 \pmod{5}$. But the last congruence has no solutions in $\mathbb{Z}/5\mathbb{Z}$. From Theorem it follows that the function f has no roots at all.

Thus, by choosing N sufficiently large, we cannot by solving a finite number of congruences modulo 5^k, $k \leq S$ determine the existence of the root of the function f.

Roots of 1-Lipschitz functions

In this regard, we need some restrictions on the 1-Lipschitz function.

- We would like to know when one can determine whether or not the function has a root and find the approximate value of this root in the p-adic metric with the given accuracy by solving a finite number of congruences.
- In other words, **under what restrictions on the 1-Lipschitz function do we get an analogue of Hensel's lifting lemma?**
- The following theorem gives such restrictions in terms of the properties of the van der Put coefficients. The next Theorem generalizes Hensel's lifting lemma for 1-Lipschitz functions.

Definitions: van der Put series

Given a continuous function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, there exists a unique sequence $B_0, B_1, B_2, \ldots$ of p-adic integers such that for all $x \in \mathbb{Z}_p$

$$f(x) = \sum_{m=0}^{\infty} B_m \chi(m,x). \tag{0.1}$$

Let $m = m_0 + \ldots + m_{n-2}p^{n-2} + m_{n-1}p^{n-1}$ be a base-p expansion for m, i.e.,
$m_j \in \{0,\ldots,p-1\}$, $j = 0,1,\ldots,n-1$ and $m_{n-1} \neq 0$, then

$$B_m = \begin{cases} f(m) - f(m - m_{n-1}p^{n-1}), & \text{if } m \geq p; \\ f(m), & \text{otherwise.} \end{cases}$$

Definitions: van der Put series

The **characteristic function** $\chi(m,x)$ is given by

$$\chi(m,x) = \begin{cases} 1, & \text{if } |x-m|_p \le p^{-n} \\ 0, & \text{otherwise} \end{cases}$$

where $n = 1$ if $m = 0$; and n is uniquely defined by the inequality $p^{n-1} \le m \le p^n - 1$ otherwise.
The number n is just the number of digits in a base-p expansion of $m \in \mathbb{N}_0$. Then
$\lfloor \log_p m \rfloor =$ (the number of digits in a base-p expansion for m) $- 1$,
therefore $n = \lfloor \log_p m \rfloor + 1$ for all $m \in \mathbb{N}_0$ and $\lfloor \log_p 0 \rfloor = 0$ ($\lfloor \alpha \rfloor$ for a real α denotes the integral part of α).

Definitions: p-adic Lipschitz functions

1-Lipschitz functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$ in terms of the van der Put series were described by W. Schikhof. But we follow notations from papers by Anashin, Khrennikov, Yurova in convenience for further study.
The function f is **1-Lipschitz** if and only if it can be represented as

$$f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x) \tag{0.2}$$

for suitable $b_m \in \mathbb{Z}_p$, $m \geq 0$.
So for every $a \in \{0, 1, \dots, p^k - 1\}$, $k \geq 1$ we set functions $\psi_a : \{0, \dots, p-1\} \to \{0, \dots, p-1\}$ defined by the relations:

$$\psi_a(i) = \begin{cases} b_{a+p^k i} (\operatorname{mod} p), & i \neq 0; \\ 0, & i = 0. \end{cases} \tag{0.3}$$

Generalization of Hensel's lifting lemma for 1-Lipschitz functions

Theorem 2. Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a 1-Lipschitz function represented via van der Put series $f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x)$.*If*

1. for some natural number R there exists $\bar{h} \in \{0, 1, \ldots, p^R - 1\}$ such that $f(\bar{h}) \equiv 0 \pmod{p^R}$ and
2. for any $m \geq R$, where $m \equiv \bar{h} \pmod{p^R}$ functions $\psi_m : \{0, \ldots, p-1\} \to \mathbb{Z}/p\mathbb{Z}$,

$$\psi_m(i) = \begin{cases} b_{m+i\cdot p^{1+\lfloor \log_p m \rfloor}} (\mathrm{mod}\, p), & i \neq 0; \\ 0, & i = 0. \end{cases}$$

   are bijective (i.e. $b_{m+i\cdot p^{1+\lfloor \log_p m \rfloor}} (\mathrm{mod}\, p)$, $i = 1, 2, \ldots, p-1$ are all nonzero residues modulo p),

then there exists a unique p-adic integer $h \in \mathbb{Z}_p$ such that $f(h) = 0$ and $h \equiv \bar{h} \pmod{p^R}$.

Roots of 1-Lipschitz functions

- From Theorem 2 follows the "classical" Hensel's lifting lemma.
- Criterion of the existance of the root in Theorem 2 does not require that functions are polynomials (as in Hensel's lifting lemma). Moreover, this Theorem does not require that the functions are differentiable.
- The proof is constructive, i.e. there is an algorithm to construct an approximate value of the root of f with given p-adic accuracy (similar to proof of Hensel's lemma). Difference: at each step we solve nonlinear congruence modulo p.
- Solution of such nonlinear congruences reduces to calculation of p values of the function by suitable modulo p^k.

Example

Let $p = 5$ and

$$f(x) = f(x_0 + 5x_1 + \ldots + 5^k \cdot x_k + \ldots) = -3 + \sum_{k=0}^{\infty} 5^k(1 + 5k^2)x_k^3.$$

We show that f has a unique root and find its approximation within 5^{-4} in 5-adic metric. Note that f is non-differentiable at any point from $\mathbb{Z}_5$. Indeed, for any $k > 1$ and $\bar{x} \in \{0, \ldots, 5^k - 1\}$

$$f(\bar{x}+5^k h) - f(\bar{x}) = 5^k(1+5k^2)h^3 \equiv 5^k h^3 \pmod{5^{k+1}}, \quad h \in \{1, 2, 3, 4\}.$$

This means that we cannot use Hensel's lifting lemma for finding the roots of f but we can use Theorem 2 instead.

Example

Let us find the coefficients of $B_m,\ m \geq 5$ with the aid of the van der Put series of the function f. Let
$m = \bar{m} + 5^k \cdot i,\ \bar{m} \in \{0, \dots, 5^k - 1\},\ i \in \{1, 2, 3, 4\},\ k \geq 1$, then

$$B_m = f(\bar{m} + 5^k \cdot i) - f(\bar{m}) = 5^k(1 + 5k^2) \cdot i^3.$$

In this notation, the functions $\psi_m,\ m \geq 5$ can be represented as

$$\psi_m(i) \equiv i^3 \pmod 5$$

Because $gcd(3, 4) = 1$, then $\psi_m,\ m \geq 5$ are bijective on $\mathbb{Z}/5\mathbb{Z}$ (i.e. the second condition of Theorem 2 holds).
Since $x \equiv x_0 = 2 \pmod 5$ is the only solution of the congruence $f(x) \equiv x_0^3 - 3 \equiv 0 \pmod 5$, then by Theorem 2 function f has a unique root in $\mathbb{Z}_5$, namely, h, and $h \equiv 2 \pmod 5$.

Example

Now we find an approximation of the root h to within 5^{-4}. If $h = h_0 + 5h_1 + \ldots + 5^k h_k + \ldots, \quad h_k \in \{0, 1, 2, 3, 4\}$, then to find such approximation it is necessary to determine the values $h_0, \; h_1, \; h_2, \; h_3$. Let us use an iterative procedure [this paper].

Value h_0 has already been defined, i.e. $h_0 = 2$. Let $h^{(s)} = h_0 + \ldots + 5^s h_s, \; s \in \{0, 1, 2, 3\}$.

Value $h_1 \in \{0, 1, 2, 3, 4\}$ we determine from the condition

$$f(h^{(0)} + 5 \cdot h_1) = f(2 + 5 \cdot h_1) \equiv 0 \pmod{5^2}.$$

By computing $f(2 + 5 \cdot i) \pmod{5^2}, \; i \in \{0, 1, 2, 3, 4\}$, we get $h_1 = 4$ and $h^{(1)} = 22$.

Value $h_2 \in \{0, 1, 2, 3, 4\}$ we determine from the condition

$$f(h^{(1)} + 5 \cdot h_2) = f(22 + 5^2 \cdot h_1) \equiv 0 \pmod{5^3}.$$

By computing $f(22 + 5 \cdot i) \pmod{5^3}, \; i \in \{0, 1, 2, 3, 4\}$, we get $h_2 = 2$ and $h^{(2)} = 72$.

Example

Value $h_3 \in \{0, 1, 2, 3, 4\}$ we determine from the condition

$$f(h^{(2)} + 5^3 \cdot h_3) = f(72 + 5^3 \cdot h_3) \equiv 0 \pmod{5^4}.$$

By computing $f(72 + 5^3 \cdot i) \pmod{5^4}$, $i \in \{0, 1, 2, 3, 4\}$, we get $h_3 = 2$ and $h^{(3)} = 322$.
Thus, the approximate value of the root of f to within 5^{-4} is 322, i.e. $|h - 322|_5 \leq 5^{-4}$.

Roots of p^α-Lipschitz functions

- Then we generalize Hensel's lifting lemma for the case of p^α-Lipschitz functions.
- The problem of finding the roots and their approximations for p^α-Lipschitz functions ($\alpha > 0$) is reduced to solving the corresponding problem for 1-Lipschitz sub-functions.
- Such possibility follows from the following Theorem.

Generalization of Hensel's lifting lemma for p^α-Lipschitz functions

Theorem 3. (representation) Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a p^α-Lipschitz function ($\alpha \geq 1$) represented as

$$f(a + p^\alpha x) = \sum_{i=0}^{p^\alpha - 1} I_i(a) \cdot f_i(x), \ a \in \{0, \ldots, p^\alpha - 1\}, \qquad (0.4)$$

where all the functions $f_i(x) : \mathbb{Z}_p \to \mathbb{Z}_p, \ i \in \{0, \ldots, p^\alpha - 1\}$ satisfy Lipschitz condition with constant 1 and

$$I_i(a) = \begin{cases} 1, & \text{if } \ a = i; \\ 0, & \text{otherwise.} \end{cases}$$

Criterion of the existance of the root

Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a p^α-Lipschitz function. 1-Lipschitz sub-functions $f_i,\ i \in \{0, \ldots, p^\alpha - 1\}$ are represented with the aid of the van der Put series $f_i(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_{i,\ m} \cdot \chi(m, x)$. If

1. for some natural number R there exists $\bar{h} \in \{0, 1, \ldots, p^{R+\alpha} - 1\}$ such that $f(\bar{h}) \equiv 0 \pmod{p^R}$;
2. for any $m \geq R$, where $m \equiv \frac{\bar{h}-s}{p^\alpha} \pmod{p^R}$, $\ s \equiv \bar{h} \pmod{p^\alpha}$ functions $\psi_{s,\ m} : \{0, \ldots, p-1\} \to \mathbb{Z}/p\mathbb{Z}$,

$$\psi_{s,\ m}(t) = \begin{cases} b_{s,\ m + t \cdot p^{1 + \lfloor \log_p m \rfloor}} \pmod{p}, & t \neq 0; \\ 0, & t = 0. \end{cases}$$

   are bijective (i.e. $b_{s,\ m + t \cdot p^{1 + \lfloor \log_p m \rfloor}} \pmod{p}$, $t = 1, 2, \ldots, p-1$ are all nonzero residues modulo p),

then there exists a unique p-adic integer $h \in \mathbb{Z}_p$ such that $f(h) = 0$ and $h \equiv \bar{h} \pmod{p^{R+\alpha}}$.

Roots of p^α-Lipschitz functions

- To check existance of the root of the p^α-Lipschitz function f, we need to check existance of the roots of 1-Lipschitz sub-functions $f_i(x), i \in \{0, \ldots, p^\alpha - 1\}$;
- Algorithm for construction of approximate root of p^α-Lipschitz function is also presented.

Roots of p^α-Lipschitz functions. Example

Let $p = 5$ and

$$f(x_0 + 5x_1 + \ldots + 5^k x_k + \ldots) = \sum_{k=1}^{\infty} 5^k (1 + 5x_0) x_k^{x_0}.$$

We find approximation of all roots of the function $f(x) - 2$ to within 5^{-3}. We represent the function $f(x) - 2$ in the form of the statement of the Theorem 3, we obtain (using the notation $\bar{x} = x_1 + 5x_2 + \ldots$)

$$\begin{aligned} f(x_0 + 5\bar{x}) - 2 = \\ &= l_0(x_0) f_0(\bar{x}) + l_1(x_0) f_1(\bar{x}) + l_2(x_0) f_2(\bar{x}) + \\ &\qquad + l_3(x_0) f_3(\bar{x}) + l_4(x_0) f_4(\bar{x}), \end{aligned}$$

where

$$f_0(\bar{x}) = -2 + \sum_{k=1}^{\infty} 5^{k-1} = -\frac{9}{4}; \qquad f_1(\bar{x}) = -2 + 6 \sum_{k=1}^{\infty} 5^{k-1} x_k;$$

$$f_2(\bar{x}) = -2 + 11 \sum_{k=1}^{\infty} 5^{k-1} x_k^2; \qquad f_3(\bar{x}) = -2 + 16 \sum_{k=1}^{\infty} 5^{k-1} x_k^3;$$

$$f_4(\bar{x}) = -2 + 21 \sum 5^{k-1} x_k^4$$

Roots of p^α-Lipschitz functions. Example

- It is easy to see that all the functions f_i satisfy the Lipschitz condition with constant 1. For each $i = 0, 1, 2, 3, 4$ we find solutions of the congruence $f_i(x_0) \equiv 0 \pmod 5$. We get $f_1(2) \equiv 0 \pmod 5$; $f_3(3) \equiv 0 \pmod 5$; and for $i = 0, 2, 4$ appropriate congruences have no solutions (this means that the numbers of the form $5\bar{x}$, $2 + 5\bar{x}$ and $4 + 5\bar{x}$ are not the roots of the function $f(x) - 2$).
- For the functions f_1 and f_3 let us verify conditions of the Theorem 3. For this we find the van der Put coefficients of the functions f_1 and f_3. Let $m = \bar{m} + 5^k t$, $\bar{m} \in \{0, \dots, 5^k - 1\}$, $t \in \{1, 2, 3, 4\}$, then

$$B_{1,\, m} = f_1(m) - f_1(\bar{m}) = 5^k(6 \cdot t), \quad B_{3,\, m} = f_3(m) - f_3(\bar{m}) = 5^k(16 \cdot t^3).$$

Roots of p^α-Lipschitz functions. Example

- Accordingly, functions $\psi_{1,\,m}$ and $\psi_{3,\,m}$ take the form

$$\psi_{1,\,m}(t) \equiv t \pmod 5, \quad \psi_{3,\,m}(t) \equiv t^3 \pmod 5$$

  and, therefore, functions $\psi_{1,\,m}$ and $\psi_{3,\,m}$ are bijective. This means that the function $f(x) - 2$ has exactly two roots. Using an iterative procedure [from this paper] and initial approximations for f_1 and f_3, we find solutions of congruences $f_1(x) \equiv 0 \pmod{5^3}$ and $f_3(x) \equiv 0 \pmod{5^3}$.

- As a result, we obtain

$$f_1(2 + 5 \cdot 3 + 5^2 \cdot 1) = f_1(42) \equiv 0 \pmod{5^3}$$
$$f_3(3 + 5 \cdot 4 + 5^2 \cdot 2) = f_3(73) \equiv 0 \pmod{5^3}.$$

  Thus, the function $f(x) - 2$ has two roots.

- The approximate value of these roots in 5-adic metric to within 5^{-3} are $1 + 5 \cdot 42 = 211$ and $3 + 5 \cdot 73 = 368$, respectively.

Conclusions

- We generalized Hensel's lifting lemma for a wider class of functions, namely, the class of p^{α}-Lipschitz functions.
- We presented an algorithm for construction of approximate root of such functions with given p-adic accuracy.
- **"Generalization of Hensel lemma: finding of roots of p-adic Lipschitz functions"** by E. Yurova Axelsson and A. Khrennikov, Journal of Number Theory, Elsevier, volume 158, p. 217-233 (January 2016)