Generalization of Hensel lemma: finding of roots of p-adic Lipschitz functions
(joint talk with Andrei Khrennikov)

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Outline

- Definitions
- Classical Hensel’s lifting lemma
- Generalization of "Hll" for 1-Lipschitz functions
- Generalization of "Hll" for $p^\alpha$-Lipschitz functions
Definitions: \textit{\(p\)-adic numbers}

- For any prime \(p \geq 2\) the \textit{\(p\)-adic norm} \(|\cdot|_p\) is defined in the following way. For every nonzero integer \(n\) let \(\text{ord}_p(n)\) be the highest power of \(p\) which divides \(n\), i.e. \(n \equiv 0 \pmod{p^{\text{ord}_p(n)}}\), \(n \neq 0 \pmod{p^{\text{ord}_p(n)+1}}\). Then we define \(|n|_p = p^{-\text{ord}_p(n)}\), \(|0|_p = 0\). For rationals \(\frac{n}{m} \in \mathbb{Q}\) we set \(|\frac{n}{m}|_p = p^{-\text{ord}_p(n)+\text{ord}_p(m)}\).

- The completion of \(\mathbb{Q}\) with respect to the \(p\)-adic metric \(\rho_p(x,y) = |x-y|_p\) is called the \textbf{field of \(p\)-adic numbers} \(\mathbb{Q}_p\). The norm satisfies the strong triangle inequality \(|x \pm y|_p \leq \max |x|_p; |y|_p\) where equality holds if \(|x|_p \neq |y|_p\).

- The set \(\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}\) is called the \textbf{set of \(p\)-adic integers}.

- Every \(x \in \mathbb{Z}_p\) can be expanded in \textbf{canonical form}, i.e. in a convergent by \(p\)-adic norm series:

\[
x = x_0 + px_1 + \ldots + p^kx_k + \ldots, \quad x_k \in \{0, 1, \ldots, p - 1\}, \quad k \geq 0.
\]

- The \textbf{ball} of radius \(p^{-r}\) with center \(a\) is the set \(B_{p^{-r}}(a) = \{x \in \mathbb{Z}_p : |x - a|_p \leq p^{-r}\} = a + p^r\mathbb{Z}_p\).
Consider functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, which satisfy the Lipschitz condition with constant $p^\alpha$, $\alpha \geq 0$ ($p^\alpha$-Lipschitz functions).

Recall that $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is $p^\alpha$-Lipschitz function if

$$|f(x) - f(y)|_p \leq p^\alpha|x - y|_p,$$

for all $x, y \in \mathbb{Z}_p$.

This condition is equivalent to that from $x \equiv y \pmod{p^k}$ follows $f(x) \equiv f(y) \pmod{p^{k-\alpha}}$ for all $k \geq 1 + \alpha$.

A class of 1-Lipschitz functions take a special place (i.e. $\alpha = 0$). For all $k \geq 1$ a 1-Lipschitz transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ the reduced mapping modulo $p^k$ is $f_{modp^k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$, $z \mapsto f(z)$ (mod $p^k$). Mapping $f_{modp^k}$ is a well-defined (the $f_{modp^k}$ does not depend on the choice of representative in the ball $z + p^k\mathbb{Z}_p$).
Hensel’s lifting lemma

The main tool for finding the roots of $p$-adic functions that map the ring of $p$-adic integers into itself, is a classical result - **Hensel’s lifting lemma**.

**Theorem (Hensel’s lifting lemma for $p$-adic case)**

Let $f(x) \in \mathbb{Z}_p[x]$ be a polynomial with integer $p$-adic coefficients and $f'(x) \in \mathbb{Z}_p[x]$ be its formal derivative. Suppose that $\bar{a} \in \mathbb{Z}_p$ is a $p$-adic integer such that $f(\bar{a}) \equiv 0 \pmod{p}$ and $f'(\bar{a}) \not\equiv 0 \pmod{p}$.

Then there exists a unique $p$-adic integer $a \in \mathbb{Z}_p$ such that $f(a) = 0$ and $a \equiv \bar{a} \pmod{p}$. 
Example. Find the solutions of \( x^3 + x^2 + 29 \equiv 0 \pmod{25} \).

Solution. Let \( f(x) = x^3 + x^2 + 29 \). We see (by inspection) that solutions of \( f(x) \equiv 0 \pmod{5} \) have \( x \equiv 3 \pmod{5} \). Because \( f'(x) = 3x^2 + 2x \) and \( f'(3) = 33 \equiv 3 \not\equiv 0 \pmod{5} \), Hensel's lemma tells us that there is a unique solution modulo 25 of the form \( 3 + 5t \), where \( t \equiv -\frac{f'(3)}{5}f(3) \pmod{5} \). Note that \( f'(3) = 3 = 2 \), because 2 is inverse to 3 mod 5. Also note that \( \frac{f(3)}{5} = \frac{65}{5} = 13 \). It follows that \( t \equiv -2 \cdot 13 \equiv 4 \pmod{5} \). We conclude that \( x \equiv 3 + 5 \cdot 4 = 23 \) is the unique solution of \( f(x) \equiv 0 \pmod{25} \).
Hensel’s lifting lemma

Hensel’s lifting lemma (Hll) for \( p \)-adic functions characterized by the following circumstances:

1. under restrictions on the function \( f \) (value of the derivative \( f \) at a given point) Hll gives the **criterion of the existence of the root** of \( p \)-adic function \( f \) on the set of \( p \)-adic integers. By solving a finite number of congruences modulo \( p \) we can determine whether \( f \) has the root in \( \mathbb{Z}_p \);

2. Hll provides an **algorithm for finding the approximate value** (in \( p \)-adic metric) of the root of \( f \) with a given accuracy. This problem is reduced to solving linear congruences modulo \( p \) at each step of the iterative procedure. Solvability of the corresponding linear congruences is determined by the restrictions that are imposed on the derivative \( f' \);

3. Hll is **applicable for \( p \)-adic functions** defined by **polynomials with integer \( p \)-adic coefficients**. Even if the function \( f \) is defined by polynomial, but not with the \( p \)-adic integer coefficients, the "formal" use of Hensel’s lifting lemma to find the approximate value of the root leads to false results.
Question arises:

- How could we find roots of $p$-adic function that maps $p$-adic integers into itself and is not defined by a polynomial with integer coefficients, for example, if $f$ is not differentiable at all?
- We solved this problem for the $p$-adic functions that satisfy the Lipschitz condition with constant $p^\alpha$, $\alpha \geq 0$. 
General criterion for the existence of the root of 1-Lipschitz functions:

**Theorem 1.** Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be 1-Lipschitz function. The function $f$ has a root if and only if the congruences $f_{modp^k}(x) \equiv 0 \pmod{p^k}$ are solvable for any $k \geq 1$, where $f_{modp^k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$, $f_{modp^k}(z) = f(z) \pmod{p^k}$.

It follows that, in general, for 1-Lipschitz function $f$ by solving a finite number of congruences mod $p^k$ it is impossible to determine whether the function $f$ has root in $\mathbb{Z}_p$, as it is done with the use of classical Hensel’s lifting lemma.
Roots of 1-Lipschitz functions. Example

Let $p = 5$ and $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ such that

$$f(x) = f(x_0 + 5x_1 + \ldots + 5^kx_k \ldots) = x + 5^N(-x_N + x_N^2 + 3),$$

where $N$ is a fixed positive integer. Note that $f$ is 1-Lipschitz function. We need to solve a finite number of congruences of the form $f_{\text{mod} 5^k}(x) \equiv 0 \pmod{5^k}$, $k \leq S$. Let $N > S$. It is clear that $f_{\text{mod} 5^k}(0) \equiv 0 \pmod{5^k}$, $k \leq S$. If a direct analogy with the Hensel’s lifting lemma is true, then we can assume that $x = 0$ is the root of the function $f$. However, the congruence $f(x) \equiv 0 \pmod{5^{N+1}}$ has no solutions.

Indeed, suppose that $h \in \{0, \ldots, 5^{N+1} - 1\}$ be a solution of this congruence and $h = \bar{h} + 5^Nh_N$. Because $f(h) \equiv f(\bar{h}) \equiv 0 \pmod{5^N}$, then $\bar{h} = 0$ and $f(h) \equiv 5^N(h_N^2 + 3) \equiv 0 \pmod{5^{N+1}}$ or $h_N^2 + 3 \equiv 0 \pmod{5}$. But the last congruence has no solutions in $\mathbb{Z}/5\mathbb{Z}$. From Theorem it follows that the function $f$ has no roots at all.

Thus, by choosing $N$ sufficiently large, we cannot by solving a finite number of congruences modulo $5^k$, $k \leq S$ determine the existence of the root of the function $f$. 
In this regard, we need some restrictions on the 1-Lipschitz function.

- We would like to know when one can determine whether or not the function has a root and find the approximate value of this root in the $p$-adic metric with the given accuracy by solving a finite number of congruences.

- In other words, **under what restrictions on the 1-Lipschitz function do we get an analogue of Hensel’s lifting lemma?**

- The following theorem gives such restrictions in terms of the properties of the van der Put coefficients. The next Theorem generalizes Hensel’s lifting lemma for 1-Lipschitz functions.
Definitions: van der Put series

Given a continuous function \( f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \), there exists a unique sequence \( B_0, B_1, B_2, \ldots \) of \( p \)-adic integers such that for all \( x \in \mathbb{Z}_p \)

\[
f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x). \tag{0.1}
\]

Let \( m = m_0 + \ldots + m_{n-2} p^{n-2} + m_{n-1} p^{n-1} \) be a base-\( p \) expansion for \( m \), i.e., \( m_j \in \{0, \ldots, p - 1\}, j = 0, 1, \ldots, n - 1 \) and \( m_{n-1} \neq 0 \), then

\[
B_m = \begin{cases} 
  f(m) - f(m - m_{n-1} p^{n-1}), & \text{if } m \geq p; \\
  f(m), & \text{otherwise.}
\end{cases}
\]
Definitions: van der Put series

The characteristic function $\chi(m, x)$ is given by

$$
\chi(m, x) = \begin{cases} 
1, & \text{if } |x - m|_p \leq p^{-n} \\
0, & \text{otherwise}
\end{cases}
$$

where $n = 1$ if $m = 0$; and $n$ is uniquely defined by the inequality $p^{n-1} \leq m \leq p^n - 1$ otherwise.

The number $n$ is just the number of digits in a base-$p$ expansion of $m \in \mathbb{N}_0$. Then

$$\lfloor \log_p m \rfloor = \text{(the number of digits in a base-$p$ expansion for $m$)} - 1,$$

therefore $n = \lfloor \log_p m \rfloor + 1$ for all $m \in \mathbb{N}_0$ and $\lfloor \log_p 0 \rfloor = 0$ ($\lfloor \alpha \rfloor$ for a real $\alpha$ denotes the integral part of $\alpha$).
1-Lipschitz functions $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ in terms of the van der Put series were described by W. Schikhof. But we follow notations from papers by Anashin, Khrennikov, Yurova in convenience for further study.

The function $f$ is **1-Lipschitz** if and only if it can be represented as

$$f(x) = \sum_{m=0}^{\infty} p^\left\lfloor \log_p m \right\rfloor b_m \chi(m, x)$$

for suitable $b_m \in \mathbb{Z}_p$, $m \geq 0$.

So for every $a \in \{0, 1, \ldots, p^k - 1\}$, $k \geq 1$ we set functions $\psi_a : \{0, \ldots, p - 1\} \rightarrow \{0, \ldots, p - 1\}$ defined by the relations:

$$\psi_a(i) = \begin{cases} b_{a + p^k i} \mod p, & i \neq 0; \\ 0, & i = 0. \end{cases}$$
Generalization of Hensel’s lifting lemma for 1-Lipschitz functions

**Theorem 2.** Let \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) be a 1-Lipschitz function represented via van der Put series \( f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x) \).

1. for some natural number \( R \) there exists \( \bar{h} \in \{0, 1, \ldots, p^R - 1\} \) such that \( f(\bar{h}) \equiv 0 \) (mod \( p^R \)) and
2. for any \( m \geq R \), where \( m \equiv \bar{h} \) (mod \( p^R \)) functions \( \psi_m : \{0, \ldots, p-1\} \to \mathbb{Z}/p\mathbb{Z} \),

\[
\psi_m(i) = \begin{cases} 
  b_{m+i \cdot p^{\lfloor \log_p m \rfloor}} \pmod{p}, & i \neq 0; \\
  0, & i = 0.
\end{cases}
\]

are bijective (i.e. \( b_{m+i \cdot p^{\lfloor \log_p m \rfloor}} \pmod{p}, i = 1, 2, \ldots, p-1 \) are all nonzero residues modulo \( p \)),

then there exists a unique \( p \)-adic integer \( h \in \mathbb{Z}_p \) such that \( f(h) = 0 \) and \( h \equiv \bar{h} \) (mod \( p^R \)).
Roots of 1-Lipschitz functions

- From Theorem 2 follows the "classical" Hensel’s lifting lemma.
- Criterion of the existance of the root in Theorem 2 does not require that functions are polynomials (as in Hensel’s lifting lemma). Moreover, this Theorem does not require that the functions are differentiable.
- The proof is constructive, i.e. there is an algorithm to construct an approximate value of the root of $f$ with given $p$-adic accuracy (similar to proof of Hensel’s lemma). Difference: at each step we solve nonlinear congruence modulo $p$.
- Solution of such nonlinear congruences reduces to calculation of $p$ values of the function by suitable modulo $p^k$. 
Example

Let $p = 5$ and

$$f(x) = f(x_0 + 5x_1 + \ldots + 5^k \cdot x_k + \ldots) = -3 + \sum_{k=0}^\infty 5^k(1 + 5k^2)x_k^3.$$ 

We show that $f$ has a unique root and find its approximation within $5^{-4}$ in $5$-adic metric. Note that $f$ is non-differentiable at any point from $\mathbb{Z}_5$. Indeed, for any $k > 1$ and $\bar{x} \in \{0, \ldots, 5^k - 1\}$

$$f(\bar{x} + 5^k h) - f(\bar{x}) = 5^k(1 + 5k^2)h^3 \equiv 5^k h^3 \pmod{5^{k+1}}, \quad h \in \{1, 2, 3, 4\}.$$ 

This means that we cannot use Hensel’s lifting lemma for finding the roots of $f$ but we can use Theorem 2 instead.
Example

Let us find the coefficients of $B_m, m \geq 5$ with the aid of the van der Put series of the function $f$. Let $m = \bar{m} + 5^k \cdot i, \bar{m} \in \{0, \ldots, 5^k - 1\}, i \in \{1, 2, 3, 4\}, k \geq 1$, then

$$B_m = f(\bar{m} + 5^k \cdot i) - f(\bar{m}) = 5^k (1 + 5k^2) \cdot i^3.$$ 

In this notation, the functions $\psi_m, m \geq 5$ can be represented as

$$\psi_m(i) \equiv i^3 \pmod{5}$$

Because $gcd(3, 4) = 1$, then $\psi_m, m \geq 5$ are bijective on $\mathbb{Z}/5\mathbb{Z}$ (i.e. the second condition of Theorem 2 holds).

Since $x \equiv x_0 = 2 \pmod{5}$ is the only solution of the congruence $f(x) \equiv x_0^3 - 3 \equiv 0 \pmod{5}$, then by Theorem 2 function $f$ has a unique root in $\mathbb{Z}_5$, namely, $h$, and $h \equiv 2 \pmod{5}$. 
Example

Now we find an approximation of the root $h$ to within $5^{-4}$. If

$$h = h_0 + 5h_1 + \ldots + 5^k h_k + \ldots,$$

$h_k \in \{0, 1, 2, 3, 4\}$, then to find such approximation it is necessary to determine the values $h_0$, $h_1$, $h_2$, $h_3$. Let us use an iterative procedure [this paper].

Value $h_0$ has already been defined, i.e. $h_0 = 2$. Let

$$h^{(s)} = h_0 + \ldots + 5^s h_s, \ s \in \{0, 1, 2, 3\}.$$ 

Value $h_1 \in \{0, 1, 2, 3, 4\}$ we determine from the condition

$$f(h^{(0)} + 5 \cdot h_1) = f(2 + 5 \cdot h_1) \equiv 0 \pmod{5^2}.$$ 

By computing $f(2 + 5 \cdot i) \pmod{5^2}$, $i \in \{0, 1, 2, 3, 4\}$, we get $h_1 = 4$ and $h^{(1)} = 22$.

Value $h_2 \in \{0, 1, 2, 3, 4\}$ we determine from the condition

$$f(h^{(1)} + 5 \cdot h_2) = f(22 + 5^2 \cdot h_1) \equiv 0 \pmod{5^3}.$$ 

By computing $f(22 + 5 \cdot i) \pmod{5^3}$, $i \in \{0, 1, 2, 3, 4\}$, we get $h_2 = 2$ and $h^{(2)} = 72$. 
Example

Value $h_3 \in \{0, 1, 2, 3, 4\}$ we determine from the condition

$$f(h^{(2)} + 5^3 \cdot h_3) = f(72 + 5^3 \cdot h_3) \equiv 0 \pmod{5^4}.$$  

By computing $f(72 + 5^3 \cdot i) \pmod{5^4}$, $i \in \{0, 1, 2, 3, 4\}$, we get $h_3 = 2$ and $h^{(3)} = 322$.

Thus, the approximate value of the root of $f$ to within $5^{-4}$ is 322, i.e. $|h - 322|_5 \leq 5^{-4}$. 

Roots of $p^\alpha$-Lipschitz functions

▶ Then we generalize Hensel’s lifting lemma for the case of $p^\alpha$-Lipschitz functions.

▶ The problem of finding the roots and their approximations for $p^\alpha$-Lipschitz functions ($\alpha > 0$) is reduced to solving the corresponding problem for 1-Lipschitz sub-functions.

▶ Such possibility follows from the following Theorem.
Generalization of Hensel’s lifting lemma for $p^\alpha$-Lipschitz functions

Theorem 3. (representation) Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a $p^\alpha$-Lipschitz function ($\alpha \geq 1$) represented as

$$f(a + p^\alpha x) = \sum_{i=0}^{p^\alpha - 1} l_i(a) \cdot f_i(x), \ a \in \{0, \ldots, p^\alpha - 1\}, \quad (0.4)$$

where all the functions $f_i(x) : \mathbb{Z}_p \to \mathbb{Z}_p, \ i \in \{0, \ldots, p^\alpha - 1\}$ satisfy Lipschitz condition with constant 1 and

$$l_i(a) = \begin{cases} 1, & \text{if } a = i; \\ 0, & \text{otherwise}. \end{cases}$$
 Criterion of the existence of the root

Let \( f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) be a \( p^\alpha \)-Lipschitz function. 1-Lipschitz sub-functions \( f_i, \ i \in \{0, \ldots, p^\alpha - 1\} \) are represented with the aid of the van der Put series \( f_i(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_i, m \cdot \chi(m, x). \) If

1. for some natural number \( R \) there exists \( \bar{h} \in \{0, 1, \ldots, p^{R+\alpha} - 1\} \) such that \( f(\bar{h}) \equiv 0 \pmod{p^R} \); 
2. for any \( m \geq R \), where \( m \equiv \frac{\bar{h} - s}{p^\alpha} \pmod{p^R} \), \( s \equiv \bar{h} \pmod{p^\alpha} \) functions \( \psi_{s, m} : \{0, \ldots, p - 1\} \rightarrow \mathbb{Z}/p\mathbb{Z}, \)

\[
\psi_{s, m}(t) = \begin{cases} 
  b_{s, m+t \cdot p^{1+\lfloor \log_p m \rfloor}} \pmod{p}, & t \neq 0; \\
  0, & t = 0.
\end{cases}
\]

are bijective (i.e. \( b_{s, m+t \cdot p^{1+\lfloor \log_p m \rfloor}} \pmod{p}, \ t = 1, 2, \ldots, p - 1 \) are all nonzero residues modulo \( p \)),

then there exists a unique \( p \)-adic integer \( h \in \mathbb{Z}_p \) such that \( f(h) = 0 \) and \( h \equiv \bar{h} \pmod{p^{R+\alpha}} \).
Roots of $p^\alpha$-Lipschitz functions

- To check existence of the root of the $p^\alpha$-Lipschitz function $f$, we need to check existence of the roots of 1-Lipschitz sub-functions $f_i(x), i \in \{0, \ldots, p^\alpha - 1\}$;
- Algorithm for construction of approximate root of $p^\alpha$-Lipschitz function is also presented.
Roots of $p^\alpha$-Lipschitz functions. Example

Let $p = 5$ and

$$f(x_0 + 5x_1 + \ldots + 5^k x_k + \ldots) = \sum_{k=1}^{\infty} 5^k (1 + 5x_0) x_k^{x_0}.$$ 

We find approximation of all roots of the function $f(x) - 2$ to within $5^{-3}$. We represent the function $f(x) - 2$ in the form of the statement of the Theorem 3, we obtain (using the notation $\bar{x} = x_1 + 5x_2 + \ldots$)

$$f(x_0 + 5\bar{x}) - 2 =$$

$$= l_0(x_0)f_0(\bar{x}) + l_1(x_0)f_1(\bar{x}) + l_2(x_0)f_2(\bar{x}) +$$

$$+ l_3(x_0)f_3(\bar{x}) + l_4(x_0)f_4(\bar{x}),$$

where

$$f_0(\bar{x}) = -2 + \sum_{k=1}^{\infty} 5^{k-1} = -\frac{9}{4}; \quad f_1(\bar{x}) = -2 + 6 \sum_{k=1}^{\infty} 5^{k-1} x_k;$$

$$f_2(\bar{x}) = -2 + 11 \sum_{k=1}^{\infty} 5^{k-1} x_k^2; \quad f_3(\bar{x}) = -2 + 16 \sum_{k=1}^{\infty} 5^{k-1} x_k^3;$$

$$f_4(\bar{x}) = -2 + 21 \sum_{k=1}^{\infty} 5^{k-1} x_k^4.$$
Roots of $p^\alpha$-Lipschitz functions. Example

- It is easy to see that all the functions $f_i$ satisfy the Lipschitz condition with constant 1. For each $i = 0, 1, 2, 3, 4$ we find solutions of the congruence $f_i(x_0) \equiv 0 \pmod{5}$. We get $f_1(2) \equiv 0 \pmod{5}$; $f_3(3) \equiv 0 \pmod{5}$; and for $i = 0, 2, 4$ appropriate congruences have no solutions (this means that the numbers of the form $5\bar{x}$, $2 + 5\bar{x}$ and $4 + 5\bar{x}$ are not the roots of the function $f(x) - 2$).

- For the functions $f_1$ and $f_3$ let us verify conditions of the Theorem 3. For this we find the van der Put coefficients of the functions $f_1$ and $f_3$. Let $m = \bar{m} + 5^k t$, $\bar{m} \in \{0, \ldots, 5^k - 1\}$, $t \in \{1, 2, 3, 4\}$, then

$$B_{1, m} = f_1(m) - f_1(\bar{m}) = 5^k(6 \cdot t), \quad B_{3, m} = f_3(m) - f_3(\bar{m}) = 5^k(16 \cdot t^3).$$
Roots of \( p^\alpha \)-Lipschitz functions. Example

Accordingly, functions \( \psi_1, m \) and \( \psi_3, m \) take the form

\[
\psi_1, m(t) \equiv t \pmod{5}, \quad \psi_3, m(t) \equiv t^3 \pmod{5}
\]

and, therefore, functions \( \psi_1, m \) and \( \psi_3, m \) are bijective. This means that the function \( f(x) - 2 \) has exactly two roots. Using an iterative procedure [from this paper] and initial approximations for \( f_1 \) and \( f_3 \), we find solutions of congruences \( f_1(x) \equiv 0 \pmod{5^3} \) and \( f_3(x) \equiv 0 \pmod{5^3} \).

As a result, we obtain

\[
\begin{align*}
f_1(2 + 5 \cdot 3 + 5^2 \cdot 1) &= f_1(42) \equiv 0 \pmod{5^3} \\
f_3(3 + 5 \cdot 4 + 5^2 \cdot 2) &= f_3(73) \equiv 0 \pmod{5^3}.
\end{align*}
\]

Thus, the function \( f(x) - 2 \) has two roots.

The approximate value of these roots in 5-adic metric to within \( 5^{-3} \) are \( 1 + 5 \cdot 42 = 211 \) and \( 3 + 5 \cdot 73 = 368 \), respectively.
Conclusions

▶ We generalized Hensel’s lifting lemma for a wider class of functions, namely, the class of $p^\alpha$-Lipschitz functions.
▶ We presented an algorithm for construction of approximate root of such functions with given $p$-adic accuracy.