# Complete integrability of geodesic motion in Sasaki-Einstein toric spaces

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# Killing forms (1)

Let $(M, g)$ be an $n$-dimensional Riemannian manifold with the metric $g$ and let $\nabla$ be its Levi-Civita connection.
The geodesic quadratic *Hamiltonian* is:

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu \,,$$

where $P_\mu$ are canonical momenta conjugate to the coordinates $x^\mu$, $P_\mu = g_{\mu\nu} \dot{x}^\nu$ with overdot denoting proper time derivative.
Let us recall that in classical mechanics a Hamiltonian system with Hamiltonian $H$ and integrals of motion $K_j$ is called *completely integrable* (or *Liouville integrable*) if it allows $n$ integrals of motion $H, K_1, \dots, K_{n-1}$ which are well-defined functions on the phase space, in involution

$$\{H, K_j\} = 0 \,, \quad \{K_j, K_k\} = 0 \,, \quad j, k = 1, \cdots, n-1 \,,$$

and functionally independent. A system is *superintegrable* if it is completely integrable and allows further functionally independent integrals of motion.

# Killing forms (2)

*Killing vector field* $K_\mu$ preserves the metric $g$ if it satisfies Killing's equation

$$\nabla_{(\mu} K_{\nu)} = 0\,.$$

Here and elsewhere a round bracket denotes a symmetrization over the indices within.
In the presence of a Killing vector, the system of a free particle admits a conserved quantity

$$K = K_\mu \dot{x}^\mu\,,$$

which commutes with the Hamiltonian in the sense of Poisson brackets:

$$\{K, H\} = 0\,.$$

# Killing forms (3)

A *Stäckel-Killing tensor* is a covariant symmetric tensor field satisfying the generalized Killing equation

$$\nabla_{(\lambda} K_{\mu_1,\ldots,\mu_r)} = 0\,.$$

The quantities

$$K_{SK} = K_{\mu_1\ldots\mu_r} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_r}\,,$$

are constants of geodesic motion.

*Killing forms* or *Killing-Yano tensors* are differential $r$-forms satisfying the equation

$$\nabla_{(\mu} f_{\nu_1)\nu_2\ldots\nu_r} = 0\,.$$

Moreover, a Killing form $\omega$ is said to be a **special Killing form** if it satisfies for some constant $c$ the additional equation

$$\nabla_X (d\omega) = cX^* \wedge \omega\,,$$

for any vector field $X$ on $M$ .

# Killing forms (4)

An important connection between these two generalizations of the Killing vectors. Given two Killing-Yano tensors ${f^{(1)}}_{\nu_1\nu_2\ldots\nu_r}$ and ${f^{(2)}}_{\nu_1\nu_2\ldots\nu_r}$ there is a Stäckel-Killing tensor of rank $2$

$$K^{(1,2)}_{\mu\nu} = {f^{(1)}}_{\mu\lambda_2\ldots\lambda_r}{f^{(2)}}_{\nu}{}^{\lambda_2\ldots\lambda_r} + {f^{(2)}}_{\mu\lambda_2\ldots\lambda_r}{f^{(1)}}_{\nu}{}^{\lambda_2\ldots\lambda_r}\,.$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing-Yano tensors.

# Sasakian geometry (1)

A $(2n-1)$-dimensional manifold $Y$ is a *contact manifold* if there exists a 1-form $\eta$ (called a contact 1-form) on $Y$ such that

$$\eta \wedge (d\eta)^{n-1} \neq 0 \,.$$

A simple and direct definition of the Sasakian structures is the following:

A compact Riemannian manifold $(Y, g)$ is **Sasakian** if and only if its metric cone $(X = C(Y) \cong \mathbb{R}_+ \times Y \,,\, \bar{g} = dr^2 + r^2\, g)$ is Kähler.

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold $(Y, g)$ is naturally isometrically embedded into the metric cone via the inclusion $Y = \{r = 1\} = \{1\} \times Y \subset C(Y)$.

# Sasakian geometry (2)

For any choice of the contact 1-form $\eta$ there exists a unique vector field $\mathbf{K}_\eta$ such that

$$\eta(\mathbf{K}_\eta) = 1\,, \quad \mathbf{K}_\eta \lrcorner d\eta = 0\,.$$

Such a vector field $\mathbf{K}$ is called the characteristic vector field or the Reeb vector field on $Y$, has unit length and, in particular, is nowhere zero. The integral curves of $\mathbf{K}$ are geodesics and the corresponding foliation $\mathcal{F}_K$ is called the Reeb foliation.
The Reeb vector field $\mathbf{K}$ lifts to a Killing vector field on $C(Y)$

$$\tilde{\mathbf{K}} \equiv \mathcal{J}\Big(r\frac{\partial}{\partial r}\Big),$$

where $\mathcal{J}$ is the complex structure on the cone manifold.

# Sasakian geometry (3)

Let $Y$ be a Sasaki-Einstein manifold of dimension $\dim_{\mathbb{R}} Y = 2n - 1$ and its Kähler cone $X = C(Y)$ is of dimension $\dim_{\mathbb{R}} X = 2n$, ($\dim_{\mathbb{C}} X = n$).
Sasaki-Einstein geometry is naturally "sandwiched" between two Kähler-Einstein geometries as follows:

Let $(Y, g)$ be a Sasaki manifold of dimension $2n - 1$. Then the following are equivalent

1. $(Y, g)$ is Sasaki-Einstein with $\mathrm{Ric}_g = 2(n - 1)g$;
2. The Kähler cone $(C(Y), \bar{g})$ is Ricci-flat $\mathrm{Ric}_{\bar{g}} = 0$;
3. The transverse Kähler structure to the Reeb foliation $\mathcal{F}_K$ is Kähler-Einstein with $\mathrm{Ric}^T = 2ng^T$.

# Hidden symmetries on Sasaki-Einstein spaces (1)

The Killing forms of the toric Sasaki-Einstein manifold $Y$ are described by the Killing forms

$$\Psi_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots, n-1.$$

Besides these Killing forms, there are $n-1$ closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (d\eta)^k \quad , \quad k = 1, \cdots, n-1.$$

# Hidden symmetries on Sasaki-Einstein spaces (2)

Moreover in the case of the Calabi-Yau cone, the holonomy is $SU(n)$ and there are *two additional* Killing forms of degree $n$. We express the volume form of the metric cone in terms of the Kähler form $\omega = \frac{1}{2}d(r^2\eta)$

$$d\mathcal{V} = \frac{1}{n!}\omega^n .$$

Here $\omega^n$ is the wedge product of $\omega$ with itself $n$ times. The volume of a Kähler manifold can be also written as

$$d\mathcal{V} = \frac{i^n}{2^n}(-1)^{n(n-1)/2}\Omega \wedge \overline{\Omega} ,$$

where $\Omega$ is the complex volume holomorphic $(n, 0)$ form of $C(Y)$. The additional (real) Killing forms are given by the real respectively the imaginary part of the complex volume form. [Semmelmann, 2003]

# Hidden symmetries on Sasaki-Einstein spaces (3)

In order to extract the corresponding additional Killing forms of the Einstein-Sasaki spaces we make use of the fact that for any $p$-form $\psi$ on the space $Y$ we can define an associated $p+1$-form $\psi^C$ on the cone $C(Y)$:

$$\psi^C := r^p dr \wedge \psi + \frac{r^{p+1}}{p+1} d\psi.$$

$\psi^C$ is parallel if and only if $\psi$ is a special Killing form with constant $c = -(p+1)$.

# Symplectic and complex approaches to Calabi-Yau cone (1)

Let $(y, \phi)$ be the symplectic coordinates on the Calabi-Yau manifold $X$ and the symplectic (Kähler) form is $\omega = dy_i \wedge d\phi_i$. If $(X, \omega)$ is toric, the standard $n$-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ acts effectively on $X$

$$\tau : \mathbb{T}^n \to Diff(X, \omega),$$

preserving the Kähler form. $\partial/\partial\phi_i$ generate the $\mathbb{T}^n$ action, $\phi_i$ being the angular coordinates along the orbit of the torus action $\phi_i \sim \phi_i + 2\pi$. $\mathbb{T}^n$-invariant Kähler metric on $X$ is

$$ds^2 = G_{ij} dy_i dy_j + G^{ij} d\phi_i d\phi_j,$$

where $G_{ij}$ is the Hessian of the symplectic potential $G(y)$ in the $y$ coordinates

$$G_{ij} = \frac{\partial^2 G}{\partial y_i \partial y_j} \quad , \quad 1 \leq i, j \leq n,$$

and $G^{ij} = (G_{ij})^{-1}$.

# Symplectic and complex approaches to Calabi-Yau cone (2)

The almost complex structure is

$$\mathcal{J} = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix}.$$

Associated to $(X, \omega, \tau)$ there is a moment map $\mu : X \to \mathbb{R}^n$

$$\mu(y, \phi) = y,$$

i.e. the projection on the action coordinates:

$$y_i = \frac{1}{2}\left\langle r^2\eta, \frac{\partial}{\partial\phi_i}\right\rangle.$$

# Symplectic and complex approaches to Calabi-Yau cone (3)

Let us write the Reeb vector in the form:

$$\tilde{K} = b_i \frac{\partial}{\partial \phi_i}.$$

In the symplectic coordinates $(y, \phi)$ we have

$$r \frac{\partial}{\partial r} = 2 y_i \frac{\partial}{\partial y_i},$$

and the components of the Reeb vector are $b_i = 2 G_{ij} y_j$ .

# Symplectic and complex approaches to Calabi-Yau cone (4)

The moment map exhibits the Kähler cone as a Lagrangian fibration over a strictly convex rational polyhedral cone $\mathcal{C} \subset \mathbb{R}^n$ by forgetting the angular coordinates $\phi_i$

$$\mathcal{C} \left\{ y \in \mathbb{R}^n | l_a(y) \geqslant 0 \quad , \quad a = 1, \ldots, d \right\} ,$$

with the linear function $l_a(y) = (y, v_a)$, where $v_a$ are the inward pointing normal vectors to the $d$ facets of the polyhedral cone. The set of vectors $\{v_a\}$

$$v_a = v_a^i \frac{\partial}{\partial \phi_i} \quad , \quad v_a^i \in \mathbb{Z},$$

is called a *toric data*.

# Symplectic and complex approaches to Calabi-Yau cone (5)

The standard complex coordinates are $w_i$ on $\mathbb{C}\backslash\{0\}$. Log complex coordinates are $z_i = \log w_i = x_i + i\phi_i$ and in these complex coordinates the metric is

$$ds^2 = F_{ij} dx_i dx_j + F_{ij} d\phi_i d\phi_j,$$

where $F_{ij}$ is the Hessian of the Kähler potential. Note also that in the complex coordinates $z_i$ the complex structures and the Kähler form are:

$$\mathcal{J} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad , \quad \omega = \begin{pmatrix} 0 & F_{ij} \\ -F_{ij} & 0 \end{pmatrix}.$$

# Symplectic and complex approaches to Calabi-Yau cone (6)

The symplectic potential $G$ and Kähler potential $F$ are related by the Legendre transform

$$F(x) = \left( y_i \frac{\partial G}{\partial y_i} - G \right) \; (y = \partial F / \partial x).$$

Therefore $F$ and $G$ are Legendre dual to each other

$$F(x) + G(y) = \sum_j \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} \text{ at } x_i = \frac{\partial G}{\partial y_i} \text{ or } y_i = \frac{\partial F}{\partial x_i}.$$

It follows from that $F_{ij} = G^{ij}$ $(y = \partial F / \partial x)$.

# Delzant construction (1)

The image of $X$ under the moment map $\mu$ is a certain kind of convex rational polytope in $\mathbb{R}^n$ called Delzant polytope.

A convex polytope $P$ in $\mathbb{R}^n$ is **Delzant** [Delzant, 1988] if

**(a)** there are $n$ edges meeting at each vertex $p$;

**(b)** the edges meeting at the vertex $p$ are rational, i.e. each edge is of the form $1 + tu_i\,, 0 \leq t \leq \infty$ where $u_i \in \mathbb{Z}^n$;

**(c)** the $u_i, \ldots, u_n$ in (b) can be chosen to be a basis of $\mathbb{Z}^n$.

Delzant construction associates to every Delzant polytope $P \subset \mathbb{R}^n$ a closed connected symplectic manifold $(M, \omega)$ together with the Hamiltonian $\mathbb{T}^n$ action and the moment map $\mu$.

# Delzant construction (2)

Using the Delzant construction the general symplectic potential has the following form in terms of the toric data:

$$G = G^{can} + G_b + h,$$

where

$$G^{can} = \frac{1}{2}\sum_a l_a(y)\log l_a(y),$$

$$G_b = \frac{1}{2}\sum_a l_b(y)\log l_b(y) - \frac{1}{2} l_\infty(y)\log l_\infty(y),$$

with $l_b(y) = (b, y)$ , $l_\infty(y) = \sum_a (v_a, y)$ and $h$ is a homogeneous degree one function of variables $y_i$

$$h = \lambda_i y_i + t,$$

$\lambda_i, t$ being some constants.

# Delzant construction (3)

For a Calabi-Yau manifold $X$, by an appropriate $SL(n, \mathbb{Z})$ transformation, it is possible to bring the normal vectors of the polyhedral cone in the form

$$\mathbf{v}_a = (1, \mathbf{w}_a).$$

The $(n, 0)$ holomorphic form of the Ricci-flat metric on the Calabi-Yau cone is

$$\Omega = e^{i\alpha}(\det F_{ij})^{1/2} dz_1 \wedge \cdots \wedge dz_n,$$

with $\alpha$ a phase space which is fixed by requiring that $\Omega$ is a closed form. The complex coordinates can be chosen such that

$$\Omega = e^{x_1 + i\phi_1} dz_1 \wedge \cdots \wedge dz_n = dw_1 \wedge \cdots \wedge dw_n / (w_2 \ldots w_n).$$

# Complete integrability on $T^{1,1}$ space (1)

One of the most familiar example of homogeneous toric Sasaki-Einstein five-dimensional manifold is the space $T^{1,1} = S^2 \times S^3$ endowed with the following metric

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,.$$

The global defined contact 1-form is

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \,.$$

The Reeb vector field $\mathbf{K}$ has the form

$$\mathbf{K} = 3\frac{\partial}{\partial\psi} \,,$$

and is easy to see that $\eta(\mathbf{K}) = 1$ .

# Complete integrability on $T^{1,1}$ space (2)

It is convenient to work with angle coordinates

$$\Phi_1 := \psi\,,$$
$$\Phi_2 := -\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{2}\psi,$$
$$\Phi_3 = -\frac{1}{2}\phi_1 - \frac{1}{2}\phi_2 + \frac{1}{2}\psi\,.$$

writing the basis for an effectively acting $\mathbb{T}^3$ in the form:

$$\mathbf{e}_i := \frac{\partial}{\partial \Phi_i}\,,$$

Considering the momentum map, we get the action coordinates

$$y^1 = \frac{1}{6}r^2(\cos\theta_1 + 1)\,,$$
$$y^2 = \frac{1}{6}r^2(\cos\theta_2 - \cos\theta_1),$$
$$y^3 = -\frac{1}{6}r^2(\cos\theta_1 + \cos\theta_2)\,.$$

# Complete integrability on $T^{1,1}$ space (3)

The Reeb vector field written in the above basis has the components

$$\mathbf{K} = (3, 3/2, 3/2)\,,$$

consistent with the determination from toric data using $Z$-minimization or $a$-maximization.

We consider the *inward pointing* normal vectors to the cone

$$\mathbf{v}_1 = [1, 1, 1]\,, \mathbf{v}_2 = [1, 0, 1]\,, \mathbf{v}_3 = [1, 0, 0]\,, \mathbf{v}_4 = [1, 1, 0]\,.$$

# Complete integrability on $T^{1,1}$ space (4)

In order to introduce the complex coordinates on conifold we need the symplectic potential $G$ which is given by the sum $G^{can} + G_b$.

Applying the Legendre transform $x_i = \frac{\partial G}{\partial y^i}$ we finally obtain the following complex coordinates

$$
\begin{aligned}
z^1 &= 3\log r + \log\sin\theta_1 + \log\sin\theta_2 + \mathrm{i}\psi\,,\\
z^2 &= \frac{3}{2}\log r + \log\sin\frac{\theta_1}{2} + \log\cos\frac{\theta_2}{2}\\
&\quad + \frac{\mathrm{i}}{2}(\psi + \phi_1 + \phi_2)\,,\\
z^3 &= \frac{3}{2}\log r + \log\sin\frac{\theta_1}{2} + \log\sin\frac{\theta_2}{2}\\
&\quad + \frac{\mathrm{i}}{2}(\psi - \phi_1 - \phi_2)\,.
\end{aligned}
$$

For the sake of simplicity in the above relations we have ignored some additive constants.

# Complete integrability on $T^{1,1}$ space (5)

The complex holomorphic volume form is

$$\Omega = \exp(z^1) dz^1 \wedge dz^2 \wedge dz^3 = r^2 dr \wedge \Psi + \frac{r^3}{3} d\Psi \,.$$

Decomposing $\Psi = \Re\Psi + \mathrm{i}\Im\Psi$, the special real Killing forms are

$$\begin{aligned} \Re\Psi = & \cos\psi \, d\theta_1 \wedge d\theta_2 + \sin\theta_2 \sin\psi \, d\theta_1 \wedge d\phi_2 \\ & - \sin\theta_1 \sin\psi \, d\theta_2 \wedge d\phi_1 \\ & - \sin\theta_1 \sin\theta_2 \cos\psi \, d\phi_1 \wedge d\phi_2 \,, \end{aligned}$$

$$\begin{aligned} \Im\Psi = & \sin\psi \, d\theta_1 \wedge d\theta_2 - \sin\theta_2 \cos\psi \, d\theta_1 \wedge d\phi_2 \\ & + \sin\theta_1 \cos\psi \, d\theta_2 \wedge d\phi_1 \\ & - \sin\theta_1 \sin\theta_2 \sin\psi \, d\phi_1 \wedge d\phi_2 \,. \end{aligned}$$

# Complete integrability on $T^{1,1}$ space (6)

Using the contact 1-form $\eta$ we get the Killing forms:

$$\begin{aligned}\Psi_1 =& \frac{1}{9}(\sin\theta_1 d\psi\wedge d\theta_1\wedge d\phi_1 + \sin\theta_2 d\psi\wedge d\theta_2\wedge d\phi_2\\ &-\cos\theta_1\sin\theta_2 d\theta_2\wedge d\phi_1\wedge d\phi_2\\ &+\cos\theta_2\sin\theta_1 d\theta_1\wedge d\phi_1\wedge d\phi_2)\,,\\ \Psi_2 =& -\frac{2}{27}\sin\theta_1\sin\theta_2 d\psi\wedge d\theta_1\wedge d\theta_2\wedge d\phi_1\wedge d\phi_2\,.\end{aligned}$$

As expected the 5-form Killing-Yano tensor $\Psi_2$ is proportional to the volume element of the space $T^{1,1}$.

# Complete integrability on $T^{1,1}$ space (7)

The conjugate momenta to the coordinates $(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ are:

$$
\begin{aligned}
P_{\theta_1} &= \frac{1}{6}\dot{\theta}_1 \,, \\
P_{\theta_2} &= \frac{1}{6}\dot{\theta}_2 \,, \\
P_{\phi_1} &= \frac{1}{6}\sin^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_1\,\dot{\psi} + \\
&\quad \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot{\phi}_2 \,, \\
P_{\phi_2} &= \frac{1}{6}\sin^2\theta_2\,\dot{\phi}_2 + \frac{1}{9}\cos^2\theta_2\,\dot{\phi}_2 \\
&\quad + \frac{1}{9}\cos\theta_2\,\dot{\psi} + \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot{\phi}_1 \,, \\
P_{\psi} &= \frac{1}{9}\dot{\psi} + \frac{1}{9}\cos\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_2\,\dot{\phi}_2 \,,
\end{aligned}
$$

# Complete integrability on $T^{1,1}$ space (8)

The conserved Hamiltonian takes the form:

$$\begin{aligned} H =& 3\left[P_{\theta_1}^2 + P_{\theta_2}^2 + \frac{1}{\sin^2\theta_1}(P_{\phi_1} - \cos\theta_1 P_\psi)^2 \right. \\ & \left. + \frac{1}{\sin^2\theta_2}(P_{\phi_2} - \cos\theta_2 P_\psi)^2\right] + \frac{9}{2}P_\psi^2 \\ =& \frac{1}{12}(\dot\theta_1^2 + \sin^2\theta_1\dot\phi_1^2 + \dot\theta_2^2 + \sin^2\theta_2\dot\phi_2^2) \\ & + \frac{1}{18}(\dot\psi + \cos\theta_1\dot\phi_1 + \cos\theta_2\dot\phi_2)^2 \,. \end{aligned}$$

Taking into account the isometries of $T^{1,1}$, the momenta $P_{\phi_1}, P_{\phi_2}$ and $P_\psi$ are conserved.

# Complete integrability on $T^{1,1}$ space (9)

Two total $SU(2)$ angular momenta are also conserved:

$$\begin{aligned}
\vec{J}_1^{\,2} =& P_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}(P_{\phi_1} - \cos\theta_1 P_\psi)^2 + P_\psi^2 \\
=& \frac{1}{36}\left[\dot{\theta}_1^2 + \sin^2\theta_1\dot{\phi}_1^2\right] + \frac{1}{81}\Big[\dot{\psi}^2 + \cos^2\theta_1\dot{\phi}_1^2 + \cos^2\theta_2\dot{\phi}_2^2 \\
& +2\cos\theta_1\,\dot{\phi}_1\dot{\psi} + 2\cos\theta_2\dot{\phi}_2\dot{\psi} + 2\cos\theta_1\cos\theta_2\dot{\phi}_1\dot{\phi}_2\Big]\,; \\
\vec{J}_2^{\,2} =& P_{\theta_2}^2 + \frac{1}{\sin^2\theta_2}(P_{\phi_2} - \cos\theta_2 P_\psi)^2 + P_\psi^2 \\
=& \frac{1}{36}\left[\dot{\theta}_2^2 + \sin^2\theta_2\dot{\phi}_2^2\right] + \frac{1}{81}\Big[\dot{\psi}^2 + \cos^2\theta_1\dot{\phi}_1^2 + \cos^2\theta_2\dot{\phi}_2^2 \\
& +2\cos\theta_1\,\dot{\phi}_1\dot{\psi} + 2\cos\theta_2\dot{\phi}_2\dot{\psi} + 2\cos\theta_1\cos\theta_2\dot{\phi}_1\dot{\phi}_2\Big].
\end{aligned}$$

# Complete integrability on $T^{1,1}$ space (10)

The first Stäckel-Killing tensor $(K1)_{\mu\nu}$ is constructed using the real part of the Killing form $\Psi$

$$
\begin{aligned}
(K1)_{\mu\nu} &= (\Re\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu} \\
&= 6\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & \sin^2\theta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

The next Stäckel-Killing tensor is constructed from the imaginary part of $\Psi$

$$(K2)_{\mu\nu} = (\Im\Psi)_{\mu\lambda}(\Im\Psi)^{\lambda}{}_{\nu}\,,$$

and we find that this tensor has the same components as $K1$.

# Complete integrability on $T^{1,1}$ space (11)

The mixed combination of $\Re\Psi$ and $\Im\Psi$ produces the Stäckel-Killing tensor

$$(K3)_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Im\Psi)^{\lambda}{}_{\nu} + (\Im\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu}\,,$$

but it proves that all components of this tensor vanish.
Finally we construct the Stäckel-Killing tensor from the Killing form $\Psi_1$

$$(K4)_{\mu\nu} = (\Psi_1)_{\mu\lambda\sigma}(\Psi_1)^{\lambda\sigma}{}_{\nu}$$
$$= \frac{4}{3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}(3+\cos^2\theta_1) & \frac{4}{3}\cos\theta_1\cos\theta_2 & \frac{4}{3}\cos\theta_1 \\ 0 & 0 & \frac{4}{3}\cos\theta_1\cos\theta_2 & \frac{1}{3}(3+\cos^2\theta_2) & \frac{4}{3}\cos\theta_2 \\ 0 & 0 & \frac{4}{3}\cos\theta_1 & \frac{4}{3}\cos\theta_2 & \frac{4}{3} \end{pmatrix}.$$

# Complete integrability on $T^{1,1}$ space (12)

Finally we investigate the integrability of the geodesic motion on $T^{1,1}$ and for this purpose we construct the Jacobian:

$$\mathcal{J} = \frac{\partial(H, P_{\phi_1}, P_{\phi_2}, P_\psi, \vec{J}_1^{\,2}, \vec{J}_2^{\,2}, K1, K4)}{\partial(\theta_1, \theta_2, \phi_1, \phi_2, \psi, \dot{\theta}_1, \dot{\theta}_2, \dot{\phi}_1, \dot{\phi}_2, \dot{\psi})}\,.$$

The rank of this Jacobian is $5$ implying the complete integrability of the geodesic motion on $T^{1,1}$. Not all constants of motion are functionally independent. We can choose the subset $(H, P_{\phi_1}, P_{\phi_2}, P_\psi, \vec{J}_1^{\,2})$ as functionally independent constants of motion and the constants of motion $\vec{J}_2^{\,2}, K1$ and $K4$ are combinations of the chosen subset of constants:

$$\frac{1}{6}K1 = 12H - \frac{2}{3}(9P_\psi)^2\,,$$

$$\frac{3}{4}K4 = 12H + \frac{2}{3}(9P_\psi)^2\,,$$

$$6\vec{J}_2^{\,2} = 2H + 3P_\psi^2 - 6\vec{J}_1^{\,2}\,.$$

# Complete integrability on $Y^{p,q}$ spaces (1)

Infinite family $Y(p,q)$ of Einstein-Sasaki metrics on $S^2 \times S^3$ provides supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y(p,q)$ of an $S^1$-fibration over $S^2 \times S^2$ with relative prime winding numbers $p$ and $q$ is topologically $S^2 \times S^3$.

Explicit local metric of the 5-dim. $Y(p,q)$ manifold is given by the line element [Gauntlett, Martelli, Sparks, Waldram, 2004]

$$
\begin{aligned}
ds^2_{ES} = & \frac{1-c\,y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{w(y)q(y)}dy^2 \\
& + \frac{q(y)}{9}(d\psi - \cos\theta\, d\phi)^2 \\
& + w(y)\left[d\alpha + \frac{ac - 2y + c\,y^2}{6(a-y^2)}[d\psi - \cos\theta\, d\phi]\right]^2 ,
\end{aligned}
$$

where

# Complete integrability on $Y^{p,q}$ spaces (2)

$$w(y) = \frac{2(a - y^2)}{1 - cy} \quad , \quad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2}$$

and $a, c$ are constants. For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$. Otherwise the constant $c$ can be rescaled by a diffeomorphism and in what follows we assume $c = 1$.

For $0 < \alpha < 1$ we can take the range of the angular coordinates $(\theta, \Phi, \Psi)$ to be $0 \leq \theta \leq 2\pi \, , 0 \leq \Phi \leq 2\pi \, , 0 \leq \Psi \leq 2\pi$. Choosing $0 < a < 1$ the roots $y_i$ of the cubic equation

$$a - 3y^2 + 2y^3 = 0 \, ,$$

are real, one negative $(y_1)$ and two positive $(y_2, y_3)$. If the smallest of the positive roots is $y_2$, one can take the range of the coordinate $y$ to be

$$y_1 \leq y \leq y_2 \, .$$

# Complete integrability on $Y^{p,q}$ spaces (3)

Sasakian $1$-form $\eta$ is:

$$\eta = -2y d\alpha + \frac{1-y}{3}(d\psi - \cos\theta d\phi)\,.$$

Basis for an effectively acting $\mathbb{T}^3$ action is

$$e_1 = \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\,,$$
$$e_2 = \frac{\partial}{\partial\phi} - \frac{l}{2}\frac{\partial}{\partial\gamma}\,,$$
$$e_3 = \frac{\partial}{\partial\gamma}\,,$$

where $l = p - q$, $\alpha \equiv \ell\gamma$ with $\ell = \ell(p,q)$ again a constant. Reeb vector has the components

$$K = \left(3, -3, -\frac{3}{2}(l + \frac{1}{3\ell})\right)$$

# Complete integrability on $Y^{p,q}$ spaces (4)

Using the momentum map we get the action coordinates

$$
\begin{aligned}
y^1 &= \frac{r^2}{6}(1-y)(1-\cos\theta)\,, \\
y^2 &= -\frac{r^2}{6}(1-y)\cos\theta + \frac{r^2}{2} l\ell y\,, \\
y^3 &= -\ell r^2 y\,.
\end{aligned}
$$

Toric data for $Y^{p,q}$

$$
\begin{aligned}
& v_1 = [1,-1,-p]\,,\ v_2 = [1,0,0]\,, \\
& v_3 = [1,-1,0]\,,\ v_4 = [1,-2,-p+q]\,.
\end{aligned}
$$

The symplectic potential in the case of $Y^{p,q}$ contains the function $h$, in contradistinction to the case of the homogeneous Sasaki-Einstein manifold $T^{1,1}$

# Complete integrability on $Y^{p,q}$ spaces (5)

However, it can be proved that one can derive a more simple expression

$$G = \sum_{A=1}^{6} \frac{1}{2} \langle v_A, y \rangle \log \langle v_A, y \rangle \ ,$$

by introducing two additional vectors $v_5$ and $v_6$

$$v_5 := K - v_1 - v_3 = \left( 1, -1, -\frac{1}{2}p + \frac{3}{2}q - \frac{1}{2\ell} \right) \ ,$$
$$v_6 := -v_2 - v_4 = (-2, 2, p - q) \, .$$

# Complete integrability on $Y^{p,q}$ spaces (6)

Using the Legendre transform, the following complex coordinates are introduced on $C(Y^{p,q})$ (additive constants being ignored)

$$z^1 = \log\left(r^3 \sin\theta \sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\right) + \mathrm{i}\psi\,,$$

$$z^2 = -\log\left(r^3 \cos^2\frac{\theta}{2} \sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\right) + \mathrm{i}(\phi - \psi)\,,$$

$$z^3 = \log\frac{r^{\frac{p(y_1-y_3)}{1-y_1}}(y-y_3)^{\frac{p(1-y_3)}{2(1-y_1)}}}{\left(\cos\frac{\theta}{2}\right)^l \sqrt{(y-y_1)^p}} + \mathrm{i}\left(\frac{l}{2}\phi - \frac{l}{2}\psi + \gamma\right)\,.$$

# Complete integrability on $Y^{p,q}$ spaces (7)

The conjugate momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$ are:

$$P_\theta = \frac{1-y}{6}\dot{\theta}\,,$$

$$P_\phi + \cos\theta P_\psi = \frac{1-y}{6}\sin^2\theta\dot{\phi}\,,$$

$$P_y = \frac{1}{6p(y)}\dot{y}\,,$$

$$P_\alpha = w(y)\left(\dot{\alpha} + f(y)\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\right)\,,$$

$$P_\psi = w(y)f(y)\dot{\alpha} + \left[\frac{q(y)}{9} + w(y)f^2(y)\right]\left(\dot{\psi} - \cos\theta\dot{\phi}\right)$$

where

$$f(y) = \frac{a - 2y + y^2}{6(a - y^2)}\,,$$

$$p(y) = \frac{w(y)q(y)}{6} = \frac{a - 3y^2 + 2y^3}{3(1-y)}\,.$$

# Complete integrability on $Y^{p,q}$ spaces (8)

Using the momenta, the conserved Hamiltonian becomes:

$$
\begin{aligned}
H =& \frac{1}{2}\Big\{ 6p(y)P_y^2 + \frac{6}{1-y}\big(P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2\big) \\
&+ \frac{1-y}{2(a-y^2)}P_\alpha^2 + \frac{9(a-y^2)}{a-3y^2+2y^3}\left(P_\psi - \frac{a-2y+y^2}{6(a-y^2)}P_\alpha\right)^2 \Big\} \\
=& \frac{1-y}{12}(\dot\theta^2 + \sin^2\theta\dot\phi^2) + \frac{\dot y^2}{12p(y)} + \frac{q(y)}{18}(\dot\psi - \cos\theta\dot\phi)^2 \\
&+ \frac{w(y)}{2}[\dot\alpha + f(y)(\dot\psi - \cos\theta\dot\phi)]^2 \,.
\end{aligned}
$$

# Complete integrability on $Y^{p,q}$ spaces (9)

From the isometry $SU(2) \times U(1) \times U(1)$ of the metric we have that the momenta $P_\phi, P_\psi$ and $P_\alpha$ are conserved. $P_\phi$ is the third component of the $SU(2)$ angular momentum and $P_\psi, P_\alpha$ are associated to the $U(1)$ factors. In addition, the total $SU(2)$ angular momentum

$$\vec{J}^{\,2} = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\psi^2$$

is also conserved.

The explicit form of the Killing-Yano tensor $\Psi_1$ is

$$\begin{aligned}\Psi_1 = &(1-y)^2 \sin\theta \, d\theta \wedge d\phi \wedge d\psi - 6dy \wedge d\alpha \wedge d\psi \\ &+ 6\cos\theta \, d\phi \wedge dy \wedge d\alpha - 6(1-y)y \sin\theta \, d\theta \wedge d\phi \wedge d\alpha \,.\end{aligned}$$

# Complete integrability on $Y^{p,q}$ spaces (10)

Let us call $\Psi$ the Killing form on $Y^{p,q}$ related to the complex holomorphic $(3,0)$ form on $C(Y^{p,q})$.
The real part of $\Psi$ is:

$$\Re\Psi = \sqrt{\frac{1-y}{p(y)}}$$
$$\times \Big(\cos\psi\Big[d\theta\wedge dy + 6p(y)\sin\theta\, d\phi\wedge d\alpha + p(y)\sin\theta\, d\phi\wedge d\psi\Big]$$
$$-\sin\psi\Big[\sin\theta\, d\phi\wedge dy - 6p(y)d\theta\wedge d\alpha - p(y)d\theta\wedge d\psi$$
$$+ p(y)\cos\theta\, d\theta\wedge d\phi\Big]\Big)\,.$$

# Complete integrability on $Y^{p,q}$ spaces (11)

The imaginary part of $\Psi$ is:

$$\Im\Psi = \sqrt{\frac{1-y}{p(y)}}$$
$$\times \Big(\sin\psi\Big[d\theta\wedge dy + 6p(y)\sin\theta\, d\phi\wedge d\alpha + p(y)\sin\theta\, d\phi\wedge d\psi\Big]$$
$$+\cos\psi\Big[\sin\theta\, d\phi\wedge dy - 6p(y)d\theta\wedge d\alpha - p(y)d\theta\wedge d\psi$$
$$+ p(y)\cos\theta\, d\theta\wedge d\phi\Big]\Big)\,.$$

# Complete integrability on $Y^{p,q}$ spaces (12)

The first Stäckel-Killing tensor $(K1)_{\mu\nu}$ is constructed using the real part of the Killing form $\Psi$:

$$(K1)_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu}$$

The corresponding conserved quantity is

$$\begin{aligned} K1 &= 6(1-y)\dot{\theta}\dot{\theta} \\ &+ \frac{3+a-6y+2y^3+(-3+a+6y-6y^2+2y^3)\cos 2\theta}{1-y}\dot{\phi}\dot{\phi} \\ &- 24\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}\dot{\phi}\dot{\alpha} - 4\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}\dot{\phi}\dot{\psi} \\ &+ 18\frac{1-y}{a+(-3+2y)y^2}\dot{y}\dot{y} + 72\frac{a+(-3+2y)y^2}{1-y}\dot{\alpha}\dot{\alpha} \\ &+ 24\frac{a+(-3+2y)y^2}{1-y}\dot{\alpha}\dot{\psi} + 2\frac{a+(-3+2y)y^2}{1-y}\dot{\psi}\dot{\psi}\,. \end{aligned}$$

# Complete integrability on $Y^{p,q}$ spaces (13)

The next Stäckel-Killing tensor will be constructed from the imaginary part of $\Psi$:

$$(K2)_{\mu\nu} = (\Im\Psi)_{\mu\lambda}(\Im\Psi)^{\lambda}{}_{\nu}\,,$$

and we find that this tensor has the same components as $K1$. The mixed combination of $\Re\Psi$ and $\Im\Psi$ produces the Stäckel-Killing tensor

$$(K3)_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Im\Psi)^{\lambda}{}_{\nu} + (\Im\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu}\,,$$

but it proves that all components of this tensor vanish.

# Complete integrability on $Y^{p,q}$ spaces (14)

Finally we construct the Stäckel-Killing tensor from the Killing form $\Psi_1$:

$$(K4)_{\mu\nu} = (\Psi_1)_{\mu\lambda\sigma}(\Psi_1)^{\lambda\sigma}{}_{\nu}\,.$$

the corresponding conserved quantity is:

$$\begin{aligned}
K4 = 18\Bigg[ & 6(1-y)\dot{\theta}\dot{\theta} - 24\frac{(a+(-4+5y-2y^2)y)\cos\theta}{1-y}\dot{\phi}\dot{\alpha} \\
& + \frac{7+a-18y+12y^2-2y^3+(1+a-6y+6y^2-2y^3)\cos 2\theta}{1-y}\dot{\phi}\dot{\phi} \\
& - 4\frac{(a-(2-y)^2(-1+2y))\cos\theta}{1-y}\dot{\phi}\dot{\psi} \\
& + 18\frac{1-y}{a+(-3+2y)y^2}\dot{y}\dot{y} + 72\frac{a+(1-2y)y^2}{1-y}\dot{\alpha}\dot{\alpha} \\
& + 24\frac{a+(-4+5y-2y^2)y}{1-y}\dot{\alpha}\dot{\psi} + 2\frac{a-(2-y)^2(-1+2y)}{1-y}\dot{\psi}\dot{\psi}\Bigg]
\end{aligned}$$

# Complete integrability on $Y^{p,q}$ spaces (15)

Having in mind that $K1 = K2$ and $K3$ vanishes, we shall verify if the set $H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2}, K1, K4$ constitutes a functionally independent set of constants of motion for the geodesics of $Y^{p,q}$ constructing the Jacobian:

$$\mathcal{J} = \frac{\partial(H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2}, K1, K4)}{\partial(\theta, \phi, y, \alpha, \psi, \dot{\theta}, \dot{\phi}, \dot{y}, \dot{\alpha}, \dot{\psi})}\,.$$

Evaluating the rank of this Jacobian we find:

$$\text{Rank}\ \mathcal{J} = 5\,,$$

which means that the system is completely integrable. In spite of the presence of the Stäckel-Killing tensors $K1$ and $K4$ the system is not superintegrable, $K1$ and $K4$ being some combinations of the first integrals $H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2}$.

# Outlook

- Hidden symmetries on higher dimensional toric Sasaki-Einstein spaces
- Hidden symmetries of other spacetime structures
- Separability of Hamilton-Jacobi, Klein-Gordon, Dirac equations