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SELF-ADJOINTNESS, CONFINEMENT AND THE CASIMIR EFFECT

Yu.A.Sitenko
BITP, Kyiv, Ukraine

Casimir effect (1948)
vacuum fluctuations of quantized electromagnetic field under the influence of boundary conditions

vacuum energy:

$$\frac{E}{S} = -\frac{\pi^2}{720}\,\frac{1}{a^3}$$

force (or pressure):

$$F = -\frac{\partial}{\partial a}\,\frac{E}{S} = -\frac{\pi^2}{240}\,\frac{1}{a^4}$$

$$a = 1\,\mu\mathrm{m} \qquad S = 100\,\mathrm{mm}^2 \qquad |F| \approx 1.3\,\mathrm{mPa}$$

Outline

- Self-adjointness and boundary conditions for confined scalar field
- Self-adjointness and boundary conditions for confined spinor field
- Confined matter in the magnetic field background
- Pressure from the vacuum of confined matter

Operator of quantized scalar field in static background field

$$\hat{\Psi}(t,\mathbf{r}) = \sum\!\!\!\!\!\!\int_{\lambda} \frac{1}{\sqrt{2\omega_\lambda}}[e^{-\mathrm{i}\omega_\lambda t}\psi_\lambda(\mathbf{r})\hat{a}_\lambda + e^{\mathrm{i}\omega_\lambda t}\psi^*_\lambda(\mathrm{r})\hat{b}^\dagger_\lambda]$$

$$[\hat{a}_\lambda, \hat{a}^\dagger_{\lambda'}]_- = [\hat{b}_\lambda, \hat{b}^\dagger_{\lambda'}]_- = \langle\lambda|\lambda'\rangle \qquad \hat{a}_\lambda|\mathrm{vac}\rangle = \hat{b}_\lambda|\mathrm{vac}\rangle = 0$$

$$(-\boldsymbol{\nabla}^2 + m^2)\psi_\lambda(\mathbf{r}) = \omega^2_\lambda\psi_\lambda(\mathbf{r})$$

Energy-momentum tensor

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_{\mathrm{can}} + \xi\nabla_\rho\hat{\underline{\Xi}}^{\mu\nu\rho}, \qquad \hat{\underline{\Xi}}^{\mu\nu\rho} = -\hat{\underline{\Xi}}^{\mu\rho\nu}$$

Temporal component

$$\hat{T}^{00} = [\partial_0\hat{\Psi}^\dagger, \partial_0\hat{\Psi}]_+ - \left[\frac{1}{4}({\partial_0}^2 - \boldsymbol{\nabla}^2) + \xi\boldsymbol{\nabla}^2\right][\hat{\Psi}^\dagger, \hat{\Psi}]_+$$

Vacuum energy density

$$\varepsilon = \langle\mathrm{vac}|\hat{T}^{00}|\mathrm{vac}\rangle = \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda\psi^*_\lambda(\mathbf{r})\psi_\lambda(\mathbf{r}) +$$

$$+\left(\frac{1}{4} - \xi\right)\boldsymbol{\nabla}^2 \sum\!\!\!\!\!\!\int_{\lambda} \omega^{-1}_\lambda\psi^*_\lambda(\mathbf{r})\psi_\lambda(\mathbf{r})$$

Self-adjointness of the Laplace operator

Defining a scalar product as $(\tilde{\chi}, \chi) = \int_D d^3r\, \tilde{\chi}^* \chi$,

we get, using integration by parts,

$$(\tilde{\chi}, \boldsymbol{\nabla}^2 \chi) = (\boldsymbol{\nabla}^2 \tilde{\chi}, \chi) + \mathrm{i} \int_{\partial D} \mathrm{d}\boldsymbol{\sigma} \cdot \mathbf{J}[\tilde{\chi}, \chi],$$

where ∂D is a two-dimensional surface bounding the three-dimensional spatial region D, $\boldsymbol{\nabla}$ is the covariant derivative involving both affine and bundle connections, and

$$\mathbf{J}[\tilde{\chi}, \chi] = -\mathrm{i}[\tilde{\chi}^* (\boldsymbol{\nabla} \chi) - (\boldsymbol{\nabla} \tilde{\chi})^* \chi].$$

The covariant Laplace operator, $\boldsymbol{\nabla}^2$, is Hermitian,

$$(\tilde{\chi}, \boldsymbol{\nabla}^2 \chi) = (\boldsymbol{\nabla}^2 \tilde{\chi}, \chi),$$

if

$$\int_{\partial D} \mathrm{d}\boldsymbol{\sigma} \cdot \mathbf{J}[\tilde{\chi}, \chi] = 0 \qquad \text{or} \qquad \mathbf{n} \cdot \mathbf{J}[\tilde{\chi}, \chi]|_{\mathbf{r} \in \partial D} = 0.$$

It is almost evident that the latter condition can be satisfied by imposing different boundary conditions for χ and $\tilde{\chi}$. But, a nontrivial task is to find a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for χ; then operator ∇^2 is self-adjoint. The spectral theorem is valid for self-adjoint operators only, and this allows one to construct appropriate unitary operator exponentials playing the key role in defining the dynamical evolution of quantum systems, see, e.g.,

J.von Neumann, *Mathematische Grundlagen der Quantummechanik* (Springer, Berlin, 1932)

Let us consider

$$\partial D : \quad \partial D^{(+)} \oplus \partial D^{(-)}$$

$$\mathbf{r} = (x, y, z) \qquad \partial D^{(+)} : \quad z = a/2; \qquad \partial D^{(-)} : \quad z = -a/2$$

.

$$
\begin{aligned}
\mathbf{n}\cdot\mathbf{J}[\tilde{\chi},\chi]|_{\mathbf{r}\in\partial D} &= J^z[\tilde{\chi},\chi]|_{z=a/2} - J^z[\tilde{\chi},\chi]|_{z=-a/2} \\
&= \frac{1}{2a}\left\{[(\tilde{\chi}+\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi+\mathrm{i}a\nabla_z\chi)]|_{z=-a/2}\right. \\
&\quad +[(\tilde{\chi}-\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi-\mathrm{i}a\nabla_z\chi)]|_{z=a/2} \\
&\quad -[(\tilde{\chi}-\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi-\mathrm{i}a\nabla_z\chi)]|_{z=-a/2} \\
&\quad \left.-[(\tilde{\chi}+\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi+\mathrm{i}a\nabla_z\chi)]|_{z=a/2}\right\}.
\end{aligned}
$$

Then $\mathbf{n}\cdot\mathbf{J}[\tilde{\chi},\chi]|_{\mathbf{r}\in\partial D} = 0$ if

$$
\begin{pmatrix} (\chi+\mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi-\mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix} = U \begin{pmatrix} (\chi-\mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi+\mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix},
$$

$$
\begin{pmatrix} (\tilde{\chi}+\mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi}-\mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix} = U \begin{pmatrix} (\tilde{\chi}-\mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi}+\mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix},
$$

where U is a $U(2)$-matrix which is in general parametrized as

$$
U = e^{-\mathrm{i}\mu}\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \qquad 0<\mu<\pi, \qquad |u|^2+|v|^2=1
$$

Explicit form of the boundary condition ensuring the self-adjointness of the Laplace operator in the case of $\partial D : \partial D^{(+)} \oplus \partial D^{(-)}$ is

$$\{[1 - e^{-\mathrm{i}\mu}(u^* \pm v)]a\nabla_z + 2\mathrm{i}[1 + e^{-\mathrm{i}\mu}(u^* \pm v)]\}\chi|_{z=a/2}$$
$$= \{\mp[1 - e^{-\mathrm{i}\mu}(u \mp v^*)]a\nabla_z \pm 2\mathrm{i}[1 + e^{-\mathrm{i}\mu}(u \mp v^*)]\}\chi|_{z=-a/2}$$

(the same condition is for $\tilde{\chi}$).

4-parametric boundary condition was discuussed in

M.Carreau, E.Farhi, S.Gutman, Phys. Rev. D 42, 1194 (1990)
G.Bonneau, J.Faraut, G.Valent, Am. J. Phys. 69, 322 (2001)

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Self-adjointness of the momentum in the normal to the boundary direction

$$(\tilde{\chi}, -\mathrm{i}\nabla_z \chi) = (-\mathrm{i}\nabla_z \tilde{\chi}, \chi) - \mathrm{i} \int\limits_{\partial D} \mathrm{d}\sigma^z \, \tilde{\chi}^* \chi$$

$$\chi|_{z=a/2} = \tilde{u}\chi|_{z=-a/2}, \quad |\tilde{u}|^2 = 1$$

1-parametric boundary condition, see, e.g.,

N.I.Akhiezer and I.M.Glazman, *Theory of Linear Operators in Hilbert Space*. Vol. 2 (Pitman, Boston, 1981)

In relativistic theory a quest is for the self-adjointness of the one-particle energy operator

$$H_{(0)} = \sqrt{-\nabla^2 + m^2}$$

Klein-Fock-Gordon equation:

$$H_{(0)}^2 \psi = \omega^2 \psi$$

1. Requirement of the continuity of the spectrum:

$$\omega^2 \geq m^2 \qquad \rightarrow \qquad (\psi, H_{(0)}^2 \psi) \geq m^2 (\psi, \psi)$$

$$0 \leq (\psi, -\nabla^2 \psi) = (-i\nabla\psi, -i\nabla\psi) - \int_{\partial D} \mathrm{d}\boldsymbol{\sigma} \cdot \psi^* (\nabla\psi)$$

$$0 \leq (-\nabla^2 \psi, \psi) = (-i\nabla\psi, -i\nabla\psi) - \int_{\partial D} \mathrm{d}\boldsymbol{\sigma} \cdot (\nabla\psi)^* \psi$$

Condition:

$$\frac{1}{2}\int_{\partial D} \mathrm{d}\boldsymbol{\sigma}\cdot\{\mathbf{J}[\tilde{\chi},\chi]\pm\mathbf{I}[\tilde{\chi},\chi]\}=0$$

$$\mathbf{J}[\tilde{\chi},\chi]=-\mathrm{i}[\tilde{\chi}^*(\boldsymbol{\nabla}\chi)-(\boldsymbol{\nabla}\tilde{\chi})^*\chi]$$

$$\mathbf{I}[\tilde{\chi},\chi]=-\mathrm{i}[\tilde{\chi}^*(\boldsymbol{\nabla}\chi)+(\boldsymbol{\nabla}\tilde{\chi})^*\chi]$$

$$\mathbf{n}\cdot\mathbf{J}[\tilde{\chi},\chi]|_{\mathbf{r}\in\partial D}=0,\qquad \mathbf{n}\cdot\mathbf{I}[\tilde{\chi},\chi]|_{\mathbf{r}\in\partial D}=0.$$

$$\begin{aligned}\mathbf{n}\cdot\mathbf{I}[\tilde{\chi},\chi]|_{\mathbf{r}\in\partial D} &= I^z[\tilde{\chi},\chi]|_{z=a/2}-I^z[\tilde{\chi},\chi]|_{z=-a/2}\\ &=\frac{1}{2a}\left\{[(\tilde{\chi}-\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi+\mathrm{i}a\nabla_z\chi)]|_{z=-a/2}\right.\\ &\quad+[(\tilde{\chi}+\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi-\mathrm{i}a\nabla_z\chi)]|_{z=a/2}\\ &\quad-[(\tilde{\chi}+\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi-\mathrm{i}a\nabla_z\chi)]|_{z=-a/2}\\ &\quad\left.-[(\tilde{\chi}-\mathrm{i}a\nabla_z\tilde{\chi})^*(\chi+\mathrm{i}a\nabla_z\chi)]|_{z=a/2}\right\}.\end{aligned}$$

$$\begin{pmatrix} (\chi + \mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi - \mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix} = \tilde{U} \begin{pmatrix} (\chi - \mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi + \mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix}$$

$$\begin{pmatrix} (\tilde{\chi} - \mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi} + \mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix} = \tilde{U} \begin{pmatrix} (\tilde{\chi} + \mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi} - \mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix}$$

$$\begin{pmatrix} (\chi + \mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi - \mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix} = \tilde{U} \begin{pmatrix} (\chi - \mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi + \mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix}$$

$$\begin{pmatrix} (\tilde{\chi} - \mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi} + \mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix} = \tilde{U} \begin{pmatrix} (\tilde{\chi} + \mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi} - \mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix}$$

and recall

$$\begin{pmatrix} (\chi + \mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi - \mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix} = U \begin{pmatrix} (\chi - \mathrm{i}a\nabla_z\chi)|_{z=-a/2} \\ (\chi + \mathrm{i}a\nabla_z\chi)|_{z=a/2} \end{pmatrix}$$

$$\begin{pmatrix} (\tilde{\chi} + \mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi} - \mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix} = U \begin{pmatrix} (\tilde{\chi} - \mathrm{i}a\nabla_z\tilde{\chi})|_{z=-a/2} \\ (\tilde{\chi} + \mathrm{i}a\nabla_z\tilde{\chi})|_{z=a/2} \end{pmatrix},$$

hence

$$\tilde{U} = U, \tilde{U} = U^\dagger \qquad \rightarrow \qquad U = U^\dagger.$$

2. Requirement of confinement:

$$j^z(\mathbf{r}) = -\mathrm{i}[\psi^*(\nabla_z\psi) - (\nabla_z\psi)^*\psi] \qquad j^z|_{z=a/2} = j^z|_{z=-a/2} = 0$$

$$\psi(\mathbf{r}) = \kappa(x,y)Z(z) \qquad Z(z) = Z^*(z)$$

$$J^z[\tilde{\chi},\chi]|_{z=\pm a/2}=0$$

$$\begin{pmatrix}(\chi-\mathrm{i}a\nabla_z\chi)|_{z=-a/2}\\ (\chi+\mathrm{i}a\nabla_z\chi)|_{z=a/2}\end{pmatrix}=U^*\begin{pmatrix}(\chi+\mathrm{i}a\nabla_z\chi)|_{z=-a/2}\\ (\chi-\mathrm{i}a\nabla_z\chi)|_{z=a/2}\end{pmatrix},$$

hence

$$U^*=U^\dagger.$$

As a result
$\mu=\pi/2$, $u^*=-u$, $v^*=-v$, and

$$U=\sigma^1\cos\rho+\sigma^3\sin\rho,\quad 0\leq\rho<2\pi,$$

where $\mathrm{Im}u=\sin\rho$, $\mathrm{Im}v=\cos\rho$.
Explicit form of the boundary condition ensuring the self-adjointness of $H_{(0)}$ in the case of $\partial D:\partial D^{(+)}\oplus\partial D^{(-)}$ is

$$\left\{\begin{array}{l}\chi|_{z=-a/2}=\tan\left(\frac{1}{2}\rho+\frac{\pi}{4}\right)\chi|_{z=a/2}\\ \nabla_z\chi|_{z=-a/2}=\cot\left(\frac{1}{2}\rho+\frac{\pi}{4}\right)\nabla_z\chi|_{z=a/2}\end{array}\right\}$$

In addition

Dirichlet boundary condition:

$$\chi|_{z=-a/2} = \chi|_{z=a/2} = 0 \qquad (U = -I)$$

and Neumann boundary condition:

$$\nabla_z \chi|_{z=-a/2} = \nabla_z \chi|_{z=a/2} = 0 \qquad (U = I).$$

Vacuum energy per unit area of the boundary surface

$$\int_{-a/2}^{a/2} \mathrm{d}z\, \varepsilon = \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda + \left(\frac{1}{4} - \xi\right) \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda^{-1} \int_{-a/2}^{a/2} \mathrm{d}z\, \boldsymbol{\nabla}^2 \psi_\lambda^*(\mathbf{r}) \psi_\lambda(\mathbf{r})$$

$$= \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda + \left(\frac{1}{4} - \xi\right) \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda^{-1} \left\{ I^z[\psi_\lambda, \psi_\lambda]|_{z=a/2} - I^z[\psi_\lambda, \psi_\lambda]|_{z=-a/2} \right\}$$

$$= \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda$$

Operator of quantized spinor field in static background field

$$\hat{\Psi}(t,\mathbf{r}) = \sum\!\!\!\!\!\!\!\int_{E_\lambda>0} \mathrm{e}^{-\mathrm{i}E_\lambda t}\psi_\lambda(\mathbf{r})\hat{a}_\lambda + \sum\!\!\!\!\!\!\!\int_{E_\lambda<0} \mathrm{e}^{-\mathrm{i}E_\lambda t}\psi_\lambda(\mathbf{r})\hat{b}^\dagger_\lambda$$

$$[\hat{a}_\lambda, \hat{a}^\dagger_{\lambda'}]_+ = [\hat{b}_\lambda, \hat{b}^\dagger_{\lambda'}]_+ = \langle\lambda|\lambda'\rangle \qquad \hat{a}_\lambda|\mathrm{vac}\rangle = \hat{b}_\lambda|\mathrm{vac}\rangle = 0$$

$$H_{(1/2)}\psi_\lambda(\mathbf{r}) = E_\lambda\psi_\lambda(\mathbf{r})$$

Temporal component of the energy-momentum tensor

$$\hat{T}^{00} = \frac{\mathrm{i}}{4}[\hat{\Psi}^\dagger(\partial_0\hat{\Psi}) - (\partial_0\hat{\Psi}^T)\hat{\Psi}^{\dagger T} - (\partial_0\hat{\Psi}^\dagger)\hat{\Psi} + \hat{\Psi}^T(\partial_0\hat{\Psi}^{\dagger T})]$$

Vacuum energy density

$$\varepsilon = \langle\mathrm{vac}|\hat{T}^{00}|\mathrm{vac}\rangle = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\lambda}|E_\lambda|\psi^\dagger_\lambda(\mathbf{r})\psi_\lambda(\mathbf{r}).$$

Self-adjointness of the Dirac operator

Defining a scalar product as $(\tilde{\chi}, \chi) = \int\limits_{\Omega} \mathrm{d}^3 r \, \tilde{\chi}^\dagger \chi$,

we get, using integration by parts,

$$(\tilde{\chi}, H_{(1/2)}\chi) = (H^\dagger_{(1/2)}\tilde{\chi}, \chi) - \mathrm{i} \int\limits_{\partial D} \mathrm{d}\boldsymbol{\sigma} \cdot \mathbf{J}[\tilde{\chi}, \chi],$$

where

$$H_{(1/2)} = H^\dagger_{(1/2)} = -\mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m$$

is the formal expression for the Dirac hamiltonian operator and

$$\mathbf{J}[\tilde{\chi}, \chi] = \tilde{\chi}^\dagger \boldsymbol{\alpha} \chi.$$

Operator $H_{(1/2)}$ is Hermitian,

$$(\tilde{\chi}, H_{(1/2)}\chi) = (H^\dagger_{(1/2)}\tilde{\chi}, \chi),$$

if

$$\int\limits_{\partial D} \mathrm{d}\boldsymbol{\sigma} \cdot \mathbf{J}[\tilde{\chi}, \chi] = 0 \qquad \text{or} \qquad \mathbf{n} \cdot \mathbf{J}[\tilde{\chi}, \chi]|_{\mathbf{r} \in \partial D} = 0.$$

To fulfill the latter condition, we impose the same boundary condition for χ and $\tilde{\chi}$ in the form

$$\chi|_{\mathbf{r}\in\partial D} = K\chi|_{\mathbf{r}\in\partial D}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial D} = K\tilde{\chi}|_{\mathbf{r}\in\partial D},$$

where K is a matrix (element of the Clifford algebra) which is determined by two conditions:

$$K^2 = I$$

and

$$K^\dagger(\boldsymbol{n}\cdot\boldsymbol{\alpha})K = -\boldsymbol{n}\cdot\boldsymbol{\alpha}.$$

It should be noted that, in addition, the following combination of χ and $\tilde{\chi}$ is also vanishing at the boundary:

$$\tilde{\chi}^\dagger(\boldsymbol{n}\cdot\boldsymbol{\alpha})K\chi|_{\mathbf{r}\in\partial D} = \tilde{\chi}^\dagger K^\dagger(\boldsymbol{n}\cdot\boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial D} = 0.$$

Using the standard representation for the Dirac matrices,

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

(σ^1, σ^2 and σ^3 are the Pauli matrices), one can get

$$K = \begin{pmatrix} 0 & \varrho^{-1} \\ \varrho & 0 \end{pmatrix},$$

where condition

$$(\boldsymbol{n} \cdot \boldsymbol{\sigma})\varrho = -\varrho^{\dagger}(\boldsymbol{n} \cdot \boldsymbol{\sigma})$$

defines ϱ as a rank-2 matrix depending on four arbitrary parameters. An explicit form for matrix K is

$$K = \frac{(1 + u^2 - v^2 - \boldsymbol{t}^2)\beta + (1 - u^2 + v^2 + \boldsymbol{t}^2)I}{2\mathrm{i}(u^2 - v^2 - \boldsymbol{t}^2)}(u\boldsymbol{n}\cdot\boldsymbol{\alpha} + v\beta\gamma^5 - \mathrm{i}\boldsymbol{t}\cdot\boldsymbol{\alpha}),$$

where $\gamma^5 = \mathrm{i}\alpha^1\alpha^2\alpha^3$, and $\boldsymbol{t} = (t^1, t^2)$ is a two-dimensional vector which is tangential to the boundary.

Using parametrization

$$u = \cosh\tilde{\vartheta}\cosh\vartheta, \quad v = \cosh\tilde{\vartheta}\sinh\vartheta\cos\theta,$$

$$t^1 = \cosh\tilde{\vartheta}\sinh\vartheta\sin\theta\cos\phi, \quad t^2 = \cosh\tilde{\vartheta}\sinh\vartheta\sin\theta\sin\phi,$$

$$-\infty < \vartheta < \infty, \quad 0 \le \tilde{\vartheta} < \infty, \quad 0 \le \theta < \pi, \quad 0 \le \phi < 2\pi \quad (15)$$

in the case of $u^2 - v^2 - \boldsymbol{t}^2 \ge 1$, one gets

$$K = \frac{\beta(1+\cosh^2\tilde{\vartheta}) - I\sinh^2\tilde{\vartheta}}{2\mathrm{i}\cosh\tilde{\vartheta}}$$

$$\times[\boldsymbol{n}\cdot\boldsymbol{\alpha}\cosh\vartheta + \beta\gamma^5\sinh\vartheta\cos\theta - \mathrm{i}(\alpha^1\cos\phi + \alpha^2\sin\phi)\sinh\vartheta\sin\theta)]$$

and

$$(I-K)\chi|_{\mathbf{r}\in\partial D} = (I-K)\tilde{\chi}|_{\mathbf{r}\in\partial D} = 0$$

is the most general extension of the standard MIT bag boundary condition (K.Johnson, 1975); the latter is obtainable at $\vartheta = \tilde{\vartheta} = \theta = \phi = 0$.

In the case of $\partial D : \partial D^{(+)} \oplus \partial D^{(-)}$, there are 8 self-adjoint extension parameters: ϑ_+, $\tilde{\vartheta}_+$, θ_+ and ϕ_+ corresponding to $\partial D^{(+)}$ and ϑ_-, $\tilde{\vartheta}_-$, θ_- and ϕ_- corresponding to $\partial D^{(-)}$. However, if some symmetry is present, then the number of self-adjoint extension parameters is diminished. For instance, if the boundary consists of two parallel planes, then the cases differing by the values of ϕ_+ or ϕ_- are physically indistinguishable, since they are related by a rotation around a normal to the boundary. To avoid this unphysical degeneracy, one has to fix

$$\theta_+ = \theta_- = 0,$$

and there remains 4 self-adjoint extension parameters: ϑ_+, $\tilde{\vartheta}_+$, ϑ_- and $\tilde{\vartheta}_-$. Operator $H_{(1/2)}$ acting on functions which are defined in the region bounded by two parallel planes is self-adjoint, if the following condition holds:

$$[I - \frac{\beta(\cosh^2 \tilde{\vartheta}_\pm + 1) - I \sinh^2 \tilde{\vartheta}_\pm}{2\mathrm{i} \cosh \tilde{\vartheta}_\pm} (\pm\alpha^z \cosh \vartheta_\pm + \beta\gamma^5 \sinh \vartheta_\pm)]$$

$$\times \chi|_{z=\pm a/2} = 0.$$

In the spin-1/2 case, any immediate physical motivation to diminish the number of the self-adjoint extension parameters seems to be lacking. In this situation one can be guided by such arguments as simplicity and unambiguity of the determination of the spectrum of k_l – z-component of the wave number vector. In particular, the condition that this spectrum be independent of the values of other components of the wave number vector yields restriction

$$\vartheta_+ = \vartheta_- = \vartheta, \quad \tilde{\vartheta}_+ = \tilde{\vartheta}_- = 0, \tag{18}$$

with resulting boundary condition

$$(I \pm \mathrm{i}\beta\alpha^z \cosh\vartheta + \mathrm{i}\gamma^5 \sinh\vartheta)\chi|_{z=\pm a/2} = 0.$$

The standard MIT bag boundary condition corresponds to $\vartheta = 0$:

$$(I \pm \mathrm{i}\beta\alpha^z)\chi|_{z=\pm a/2} = 0.$$

Vacuum energy per unit area of the boundary surface

scalar case:

$$\frac{E}{S} = \int\limits_{-a/2}^{a/2} \mathrm{d}z\,\varepsilon = \sum\!\!\!\!\!\!\int_{\lambda} \omega_\lambda,$$

where

$$\sin\left[\frac{1}{2}(k_l a + \rho)\right] = 0 \quad (-\infty < k_l < \infty), \quad \rho \neq \pi/2,\, 3\pi/2$$

or

$$\cos(k_l a) = 0 \quad (0 < k_l < \infty), \quad \rho = \pi/2,\, 3\pi/2.$$

spinor case:

$$\frac{E}{S} = \int\limits_{-a/2}^{a/2} \mathrm{d}z\,\varepsilon = -\frac{1}{2}\sum\!\!\!\!\!\!\int_{\lambda} |E_\lambda|,$$

where

$$\sin\left[k_l a + \arctan\left(\frac{\hbar k_l \cosh\vartheta}{mc}\right)\right] = 0 \quad (0 < k_l < \infty).$$

Background: uniform magnetic field orthogonal to the boundary

$$\boldsymbol{B} = (0, 0, B), \qquad \boldsymbol{A} = (-yB, 0, 0)$$

$$\partial D: \quad \partial D^{(+)} \oplus \partial D^{(-)}$$

$$\mathbf{r} = (x, y, z) \qquad \partial D^{(+)}: \quad z = a/2; \qquad \partial D^{(-)}: \quad z = -a/2$$

.

$$\boldsymbol{\nabla}\hat{\Psi} = (\boldsymbol{\partial} - \mathrm{i}e\mathbf{A})\hat{\Psi}, \quad \boldsymbol{\nabla}\hat{\Psi}^\dagger = (\boldsymbol{\partial} + \mathrm{i}e\mathbf{A})\hat{\Psi}^\dagger, \quad \mathbf{B} = \boldsymbol{\partial} \times \mathbf{A}$$

One-particle energy spectrum (Landau levels):

$$\omega_{snl} = \sqrt{|eB|(2n + 1 - 2s) + k_l^2 + m^2},$$

$$s = 0, 1/2, \quad n = 0, 1, 2, ...,$$

Vacuum energy per unit area of the boundary surface

$$\frac{E_{(s)}}{S} = \frac{|eB|}{2\pi}(1-4s)\sum_{l}\sum_{n=0}^{\infty}(1+2s-2s\delta_{n0})\omega_{snl}$$

Abel-Plana summation formula and its generalizations, see

S.Bellucci and A.A.Saharian, *Phys. Rev. D* **80**, 105003 (2009).
S.Bellucci, A.A.Saharian and V.M.Bardeghyan, *Phys. Rev. D* **82**, 065011 (2010).

$$\frac{E_{(s)}}{S} = a\,\varepsilon_{(s)}^{\infty} + \Omega_{(s)}(a) + \tilde{\Omega}_{(s)},$$

where

$$\varepsilon_{(s)}^{\infty} = \frac{|eB|}{(2\pi)^2}(1-4s)\int_{-\infty}^{\infty}\mathrm{d}k\sum_{n=0}^{\infty}(1+2s-2s\delta_{n0})\omega_{snk}.$$

Regularization & renormalization: $\varepsilon^{\infty}_{(s)} \to \varepsilon^{\infty}_{(s)\mathrm{ren}}$

$$\varepsilon^{\infty}_{(s)\mathrm{ren}} = \frac{e^2 B^2}{(4\pi)^2} \int\limits_0^{\infty} \frac{\mathrm{d}\eta}{\eta} \exp\left(-\frac{m^2 \eta}{|eB|}\right) \left[\frac{4s\cosh\eta - 1 + 2s}{\eta \sinh\eta} \right.$$

$$\left. + (1-6s)\frac{1}{\eta^2} - \frac{1}{6}(1+6s)\right]$$

V.S.Weisskopf, *Kong. Dans. Vid. Selsk. Mat-Fys. Medd.* **14**, 6 (1936).
W.Heisenberg and H.Euler, *Z. Phys.* **98**, 714 (1936).

Regularization & renormalization: $\frac{E_{(s)}}{S} \to \frac{E_{(s)\mathrm{ren}}}{S}$

$$\frac{E_{(s)\mathrm{ren}}}{S} = a\,\varepsilon^{\infty}_{(s)\mathrm{ren}} + \Omega_{(s)}(a) + \tilde{\Omega}_{(s)}$$

Casimir force (or pressure)

$$F_{(s)} \equiv -\frac{\partial}{\partial a}\frac{E_{(s)\mathrm{ren}}}{S} = -\varepsilon^{\infty}_{(s)\mathrm{ren}} + \Delta_{(s)}(a),$$

where

$$\Delta_{(s)}(a) \equiv -\frac{\partial}{\partial a}\Omega_{(s)}(a)$$

$$= -\frac{|eB|}{\pi^2}\sum_{n=0}^{\infty}(1+2s-2s\delta_{n0})\int\limits_{M_{sn}}^{\infty} \mathrm{d}\kappa\,\Upsilon_{(s)}(\kappa)\kappa^{2-4s}(\kappa^2 - M_{sn}^2)^{2s-1/2},$$

$$M_{sn} = \sqrt{|eB|(2n+1-2s)+m^2},$$

$$\Upsilon_{(0)}(\kappa) = \frac{1}{2}\,\frac{\cos\rho - \mathrm{e}^{-\kappa a}}{\cosh(\kappa a) - \cos\rho},$$

$$\Upsilon_{(1/2)}(\kappa) = \frac{\left[(2\kappa a - 1)\left(\kappa^2\cosh^2\vartheta - m^2\right) - 2\kappa m\cosh\vartheta\right]\mathrm{e}^{2\kappa a}}{\left[(\kappa\cosh\vartheta + m)\,\mathrm{e}^{2\kappa a} + \kappa\cosh\vartheta - m\right]^2}$$

$$-\frac{(\kappa\cosh\vartheta - m)^2}{\left[(\kappa\cosh\vartheta + m)\,\mathrm{e}^{2\kappa a} + \kappa\cosh\vartheta - m\right]^2}.$$

$$F_{(s)} = -\varepsilon^{\infty}_{(s)\mathrm{ren}} + \Delta_{(s)}(a),$$

$-\varepsilon^{\infty}_{(s)\mathrm{ren}}$ is positive

In the case of a weak magnetic field, $|B| \ll m^2|e|^{-1}$, one has

$$-\varepsilon^{\infty}_{(s)\mathrm{ren}} = \frac{1}{360\pi^2}\left[1 - \frac{9}{8}\left(\frac{1}{2} - s\right)\right]\left(\frac{eB}{m}\right)^4 .$$

Note that the critical value is the lowest one, $B_{\mathrm{crit}} = m^2|e|^{-1} = 4.41 \times 10^{13}\,\mathrm{Gauss}$, for the case of quantized electron-positron matter.

In the case of a strong magnetic field, $|B| \gg m^2|e|^{-1}$, one has

$$-\varepsilon^{\infty}_{(s)\mathrm{ren}} = \frac{1}{24\pi^2}\left[1 - \frac{3}{2}\left(\frac{1}{2} - s\right)\right] e^2 B^2 \ln \frac{2|eB|}{m^2}.$$

$\Delta_{(1/2)}(a)$ at $|B| \ll m^2|e|^{-1}$ takes the forms in the limits of large and small distances between the plates

$$\Delta_{(1/2)}(a) = \left\{ \begin{array}{ll} -\frac{3}{16\pi^{3/2}}\frac{m^{3/2}}{a^{5/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \vartheta = 0 \\ \\ -\frac{\tanh^2(\vartheta/2)}{2\pi^{3/2}}\frac{m^{5/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \vartheta \neq 0 \end{array} \right\},$$

$$|eB| \ll m^2, \qquad ma \gg 1$$

and

$$\Delta_{(1/2)}(a) = -\frac{7}{8}\frac{\pi^2}{120}\frac{1}{a^4}, \quad |eB| \ll m^2, \qquad ma \ll 1.$$

$\Delta_{(1/2)}(a)$ at $|B| \gg m^2|e|^{-1}$ takes the forms in the limits of large and small distances between the plates

$$\Delta_{(1/2)}(a) = \left\{ \begin{array}{ll} -\frac{|eB|}{16\pi^{3/2}}\frac{m^{1/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \vartheta = 0 \\ & \\ -\frac{|eB|\tanh^2(\vartheta/2)}{2\pi^{3/2}}\frac{m^{3/2}}{a^{1/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \vartheta \neq 0 \end{array} \right\},$$

$$\sqrt{|eB|}a \gg ma \gg 1$$

and

$$\Delta_{(1/2)}(a) = -\frac{|eB|}{48a^2}, \quad ma \ll 1, \ \sqrt{|eB|}a \gg 1.$$

$$m^{-1} = 3.86 \times 10^{-13}\,\mathrm{m}, \quad \mathrm{a} > 10^{-8}\,\mathrm{m}$$

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Conclusion

- The pressure from the vacuum of confined charged massive matter in the background of a magnetic field orthogonal to plates is positive, being independent of the choice of a boundary condition, as well as of the distance between the plates.

Thank you for your attention!