SELF-ADJOINTNESS, CONFINEMENT AND THE CASIMIR EFFECT

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Casimir effect (1948)
vacuum fluctuations of quantized electromagnetic field under the influence of boundary conditions

vacuum energy:

\[
\frac{E}{S} = -\frac{\pi^2}{720} \frac{1}{a^3}
\]

force (or pressure):

\[
F = -\frac{\partial}{\partial a} \frac{E}{S} = -\frac{\pi^2}{240} \frac{1}{a^4}
\]

\[
a = 1 \mu m \quad S = 100 \text{ mm}^2 \quad |F| \approx 1.3 \text{ mPa}
\]
Self-adjointness and boundary conditions for confined scalar field
Self-adjointness and boundary conditions for confined spinor field
Confined matter in the magnetic field background
Pressure from the vacuum of confined matter
Operator of quantized scalar field in static background field

\[ \hat{\psi}(t, \mathbf{r}) = \sum_{\lambda} \int \frac{1}{\sqrt{2\omega_{\lambda}}} \left[ e^{-i\omega_{\lambda}t} \psi_{\lambda}(\mathbf{r}) \hat{a}_{\lambda} + e^{i\omega_{\lambda}t} \psi^*_{\lambda}(\mathbf{r}) \hat{b}^\dagger_{\lambda} \right] \]

\[ [\hat{a}_{\lambda}, \hat{a}^\dagger_{\lambda'}]_- = [\hat{b}_{\lambda}, \hat{b}^\dagger_{\lambda'}]_- = \langle \lambda | \lambda' \rangle \quad \hat{a}_{\lambda} | \text{vac} \rangle = \hat{b}_{\lambda} | \text{vac} \rangle = 0 \]

\[ (-\nabla^2 + m^2)\psi_{\lambda}(\mathbf{r}) = \omega^2_{\lambda}\psi_{\lambda}(\mathbf{r}) \]

Energy-momentum tensor

\[ \hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_{\text{can}} + \xi \nabla_{\rho} \hat{T}^{\mu\nu\rho} \quad \hat{T}^{\mu\nu\rho} = -\hat{T}^{\mu\rho\nu} \]

Temporal component

\[ \hat{T}^{00} = [\partial_0 \hat{\psi}^\dagger, \partial_0 \hat{\psi}]_+ - \left[ \frac{1}{4} (\partial_0^2 - \nabla^2) + \xi \nabla^2 \right] [\hat{\psi}^\dagger, \hat{\psi}]_+ \]

Vacuum energy density

\[ \varepsilon = \langle \text{vac} | \hat{T}^{00} | \text{vac} \rangle = \sum_{\lambda} \omega_{\lambda} \psi^*_{\lambda}(\mathbf{r}) \psi_{\lambda}(\mathbf{r}) + \]

\[ + \left( \frac{1}{4} - \xi \right) \nabla^2 \sum_{\lambda} \omega^{-1}_{\lambda} \psi^*_{\lambda}(\mathbf{r}) \psi_{\lambda}(\mathbf{r}) \]
Self-adjointness of the Laplace operator

Defining a scalar product as \((\tilde{\chi}, \chi) = \int_D d^3r \tilde{\chi}^* \chi\),
we get, using integration by parts,

\[(\tilde{\chi}, \nabla^2 \chi) = (\nabla^2 \tilde{\chi}, \chi) + i \int_{\partial D} d\sigma \cdot \mathbf{J}[\tilde{\chi}, \chi],\]

where \(\partial D\) is a two-dimensional surface bounding the three-dimensional spatial region \(D\), \(\nabla\) is the covariant derivative involving both affine and bundle connections, and

\[\mathbf{J}[\tilde{\chi}, \chi] = -i[\tilde{\chi}^*(\nabla \chi) - (\nabla \tilde{\chi})^* \chi].\]

The covariant Laplace operator, \(\nabla^2\), is Hermitian,

\[(\tilde{\chi}, \nabla^2 \chi) = (\nabla^2 \tilde{\chi}, \chi),\]

if

\[\int_{\partial D} d\sigma \cdot \mathbf{J}[\tilde{\chi}, \chi] = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{J}[\tilde{\chi}, \chi]|_{r \in \partial D} = 0.\]
It is almost evident that the latter condition can be satisfied by imposing different boundary conditions for $\chi$ and $\tilde{\chi}$. But, a nontrivial task is to find a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for $\chi$; then operator $\nabla^2$ is self-adjoint. The spectral theorem is valid for self-adjoint operators only, and this allows one to construct appropriate unitary operator exponentials playing the key role in defining the dynamical evolution of quantum systems, see, e.g.,


Let us consider

$$\partial D: \quad \partial D^{(+)} \oplus \partial D^{(-)}$$

$$\mathbf{r} = (x, y, z) \quad \partial D^{(+)}: \quad z = a/2; \quad \partial D^{(-)}: \quad z = -a/2.$$
\[ n \cdot J[\tilde{x}, \chi]|_{r \in \partial D} = J^z[\tilde{x}, \chi]|_{z=a/2} - J^z[\tilde{x}, \chi]|_{z=-a/2} \]
\[ = \frac{1}{2a} \left\{ \left[ (\tilde{x} + ia\nabla_z \tilde{x})^* (\chi + ia\nabla_z \chi) \right] |_{z=-a/2} + \left[ (\tilde{x} - ia\nabla_z \tilde{x})^* (\chi - ia\nabla_z \chi) \right] |_{z=a/2} \right. \]
\[ - \left[ (\tilde{x} - ia\nabla_z \tilde{x})^* (\chi - ia\nabla_z \chi) \right] |_{z=-a/2} \left. - \left[ (\tilde{x} + ia\nabla_z \tilde{x})^* (\chi + ia\nabla_z \chi) \right] |_{z=a/2} \right\}. \]

Then \( n \cdot J[\tilde{x}, \chi]|_{r \in \partial D} = 0 \) if
\[ \begin{pmatrix} (\chi + ia\nabla_z \chi)|_{z=-a/2} \\ (\chi - ia\nabla_z \chi)|_{z=a/2} \end{pmatrix} = U \begin{pmatrix} (\chi - ia\nabla_z \chi)|_{z=-a/2} \\ (\chi + ia\nabla_z \chi)|_{z=a/2} \end{pmatrix}, \]
\[ \begin{pmatrix} (\tilde{x} + ia\nabla_z \tilde{x})|_{z=-a/2} \\ (\tilde{x} - ia\nabla_z \tilde{x})|_{z=a/2} \end{pmatrix} = U \begin{pmatrix} (\tilde{x} - ia\nabla_z \tilde{x})|_{z=-a/2} \\ (\tilde{x} + ia\nabla_z \tilde{x})|_{z=a/2} \end{pmatrix}, \]
where \( U \) is a \( U(2) \)-matrix which is in general parametrized as
\[ U = e^{-i\mu} \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \quad 0 < \mu < \pi, \quad |u|^2 + |v|^2 = 1 \]
Explicit form of the boundary condition ensuring the self-adjointness of the Laplace operator in the case of $\partial D : \partial D^{(+)} \oplus \partial D^{(-)}$ is

$$\{[1 - e^{-i\mu}(u^* \pm v)]a\nabla_z + 2i[1 + e^{-i\mu}(u^* \pm v)]\}\chi|_{z=a/2}$$

$$= \{\mp[1 - e^{-i\mu}(u \mp v^*)]a\nabla_z \pm 2i[1 + e^{-i\mu}(u \mp v^*)]\}\chi|_{z=-a/2}$$

(the same condition is for $\bar{\chi}$).

4-parametric boundary condition was discussed in

Self-adjointness of the momentum in the normal to the boundary direction

\[
(\tilde{\chi}, -i\nabla_z \chi) = (-i\nabla_z \tilde{\chi}, \chi) - i \int_{\partial D} d\sigma^z \tilde{\chi}^* \chi
\]

\[
\chi|_{z=a/2} = \tilde{u} \chi|_{z=-a/2}, \quad |\tilde{u}|^2 = 1
\]

1-parametric boundary condition, see, e.g.,

In relativistic theory a quest is for the self-adjointness of the one-particle energy operator

\[ H(0) = \sqrt{-\nabla^2 + m^2} \]

Klein-Fock-Gordon equation:

\[ H^2(0) \psi = \omega^2 \psi \]

1. Requirement of the continuity of the spectrum:

\[ \omega^2 \geq m^2 \quad \rightarrow \quad (\psi, H^2(0)\psi) \geq m^2(\psi, \psi) \]

\[ 0 \leq (\psi, -\nabla^2 \psi) = (-i\nabla \psi, -i\nabla \psi) - \int_{\partial D} d\sigma \cdot \psi^* (\nabla \psi) \]

\[ 0 \leq (-\nabla^2 \psi, \psi) = (-i\nabla \psi, -i\nabla \psi) - \int_{\partial D} d\sigma \cdot (\nabla \psi)^* \psi \]
Condition:

\[ \frac{1}{2} \int_{\partial D} d\sigma \cdot \{ J[\tilde{\chi}, \chi] \pm I[\tilde{\chi}, \chi] \} = 0 \]

\[ J[\tilde{\chi}, \chi] = -i[\tilde{\chi}^*(\nabla \chi) - (\nabla \tilde{\chi})^* \chi] \]

\[ I[\tilde{\chi}, \chi] = -i[\tilde{\chi}^*(\nabla \chi) + (\nabla \tilde{\chi})^* \chi] \]

\[ n \cdot J[\tilde{\chi}, \chi]|_{r \in \partial D} = 0, \quad n \cdot I[\tilde{\chi}, \chi]|_{r \in \partial D} = 0. \]

\[ n \cdot I[\tilde{\chi}, \chi]|_{r \in \partial D} = I^2[\tilde{\chi}, \chi]|_{z=a/2} - I^2[\tilde{\chi}, \chi]|_{z=-a/2} \]
\[ = \frac{1}{2a} \left\{ [(\tilde{\chi} - ia\nabla_z \tilde{\chi})^*(\chi + ia\nabla_z \chi)]|_{z=-a/2} \right. \]
\[ + [(\tilde{\chi} + ia\nabla_z \tilde{\chi})^*(\chi - ia\nabla_z \chi)]|_{z=a/2} \]
\[ - [(\tilde{\chi} + ia\nabla_z \tilde{\chi})^*(\chi - ia\nabla_z \chi)]|_{z=-a/2} \]
\[ - [(\tilde{\chi} - ia\nabla_z \tilde{\chi})^*(\chi + ia\nabla_z \chi)]|_{z=a/2} \right\}. \]
\[
\left(
\begin{array}{c}
(\chi + i a \nabla_z \chi)|_{z=-a/2} \\
(\chi - i a \nabla_z \chi)|_{z=a/2}
\end{array}
\right) = \tilde{U} \left(
\begin{array}{c}
(\chi - i a \nabla_z \chi)|_{z=-a/2} \\
(\chi + i a \nabla_z \chi)|_{z=a/2}
\end{array}
\right)
\]

\[
\left(
\begin{array}{c}
(\tilde{\chi} - i a \nabla_z \tilde{\chi})|_{z=-a/2} \\
(\tilde{\chi} + i a \nabla_z \tilde{\chi})|_{z=a/2}
\end{array}
\right) = \tilde{U} \left(
\begin{array}{c}
(\tilde{\chi} + i a \nabla_z \tilde{\chi})|_{z=-a/2} \\
(\tilde{\chi} - i a \nabla_z \tilde{\chi})|_{z=a/2}
\end{array}
\right)
\]
\[
\begin{align*}
\left((\chi + ia\nabla_z \chi)|_{z=-a/2}\right) &= \tilde{\mathcal{U}} \left((\chi - ia\nabla_z \chi)|_{z=-a/2}\right) \\
\left((\chi - ia\nabla_z \chi)|_{z=a/2}\right) &= \tilde{\mathcal{U}} \left((\chi + ia\nabla_z \chi)|_{z=a/2}\right) \\
\left((\tilde{\chi} - ia\nabla_z \tilde{\chi})|_{z=-a/2}\right) &= \tilde{\mathcal{U}} \left((\tilde{\chi} + ia\nabla_z \tilde{\chi})|_{z=-a/2}\right) \\
\left((\tilde{\chi} + ia\nabla_z \tilde{\chi})|_{z=a/2}\right) &= \tilde{\mathcal{U}} \left((\tilde{\chi} - ia\nabla_z \tilde{\chi})|_{z=a/2}\right),
\end{align*}
\]

and recall
\[
\begin{align*}
\left((\chi + ia\nabla_z \chi)|_{z=-a/2}\right) &= \mathcal{U} \left((\chi - ia\nabla_z \chi)|_{z=-a/2}\right) \\
\left((\chi - ia\nabla_z \chi)|_{z=a/2}\right) &= \mathcal{U} \left((\chi + ia\nabla_z \chi)|_{z=a/2}\right) \\
\left((\tilde{\chi} + ia\nabla_z \tilde{\chi})|_{z=-a/2}\right) &= \mathcal{U} \left((\tilde{\chi} - ia\nabla_z \tilde{\chi})|_{z=-a/2}\right) \\
\left((\tilde{\chi} - ia\nabla_z \tilde{\chi})|_{z=a/2}\right) &= \mathcal{U} \left((\tilde{\chi} + ia\nabla_z \tilde{\chi})|_{z=a/2}\right),
\end{align*}
\]

hence
\[
\tilde{\mathcal{U}} = \mathcal{U}, \quad \tilde{\mathcal{U}} = U^\dagger \quad \rightarrow \quad U = U^\dagger.
\]

2. Requirement of confinement:

\[
\begin{align*}
\mathcal{J}^Z(r) &= -i[\psi^*(\nabla_z \psi) - (\nabla_z \psi)^* \psi] \\
\mathcal{J}^Z|_{z=a/2} &= \mathcal{J}^Z|_{z=-a/2} = 0 \\
\psi(r) &= \kappa(x, y)Z(z) \\
Z(z) &= Z^*(z)
\end{align*}
\]
\[ J^Z[\tilde{\chi}, \chi]|_{z=\pm a/2} = 0 \]

\[
\begin{pmatrix}
(\chi - i a \nabla_z \chi)|_{z=-a/2} \\
(\chi + i a \nabla_z \chi)|_{z=a/2}
\end{pmatrix}
= U^* \begin{pmatrix}
(\chi + i a \nabla_z \chi)|_{z=-a/2} \\
(\chi - i a \nabla_z \chi)|_{z=a/2}
\end{pmatrix},
\]

hence

\[ U^* = U^\dagger. \]

As a result
\[ \mu = \pi/2, \ u^* = -u, \ v^* = -v, \text{ and} \]

\[ U = \sigma^1 \cos \rho + \sigma^3 \sin \rho, \quad 0 \leq \rho < 2\pi, \]

where \(\text{Im} u = \sin \rho, \text{Im} v = \cos \rho.\)

Explicit form of the boundary condition ensuring the
self-adjointness of \(H_{(0)}\) in the case of \(\partial D: \partial D^{(+)} \oplus \partial D^{(-)}\) is

\[
\begin{cases}
\chi|_{z=-a/2} = \tan \left(\frac{1}{2} \rho + \frac{\pi}{4}\right) \chi|_{z=a/2} \\
\nabla_z \chi|_{z=-a/2} = \cot \left(\frac{1}{2} \rho + \frac{\pi}{4}\right) \nabla_z \chi|_{z=a/2}
\end{cases}
\]
In addition

Dirichlet boundary condition:

\[ \chi|_{z=-a/2} = \chi|_{z=a/2} = 0 \quad (U = -I) \]

and Neumann boundary condition:

\[ \nabla z \chi|_{z=-a/2} = \nabla z \chi|_{z=a/2} = 0 \quad (U = I). \]

Vacuum energy per unit area of the boundary surface

\[
\int_{-a/2}^{a/2} dz \varepsilon = \sum_{\lambda} \int_{-a/2}^{a/2} dz \left( \frac{1}{4} - \xi \right) \sum_{\lambda} \omega^{-1}_{\lambda} \int_{-a/2}^{a/2} dz \nabla^2 \psi^*_\lambda(r) \psi_\lambda(r) \\
= \sum_{\lambda} \omega_\lambda + \left( \frac{1}{4} - \xi \right) \sum_{\lambda} \omega^{-1}_{\lambda} \left\{ I^Z[\psi_\lambda, \psi_\lambda]|_{z=a/2} - I^Z[\psi_\lambda, \psi_\lambda]|_{z=-a/2} \right\} \\
= \sum_{\lambda} \omega_\lambda
\]
Operator of quantized spinor field in static background field

\[ \hat{\psi}(t, r) = \sum_{E_\lambda > 0} e^{-iE_\lambda t} \psi_\lambda(r) \hat{a}_\lambda + \sum_{E_\lambda < 0} e^{-iE_\lambda t} \psi_\lambda(r) \hat{b}_\lambda \]

\[ [\hat{a}_\lambda, \hat{a}_\lambda^\dagger]_+ = [\hat{b}_\lambda, \hat{b}_\lambda^\dagger]_+ = \langle \lambda | \lambda' \rangle \quad \hat{a}_\lambda |\text{vac}\rangle = \hat{b}_\lambda |\text{vac}\rangle = 0 \]

\[ H_{(1/2)} \psi_\lambda(r) = E_\lambda \psi_\lambda(r) \]

Temporal component of the energy-momentum tensor

\[ \hat{T}^{00} = \frac{i}{4} [\hat{\psi}^\dagger (\partial_0 \hat{\psi}) - (\partial_0 \hat{\psi}^T) \hat{\psi}^{\dagger T} - (\partial_0 \hat{\psi}^\dagger) \hat{\psi}^\dagger + \hat{\psi}^T (\partial_0 \hat{\psi}^{\dagger T})] \]

Vacuum energy density

\[ \varepsilon = \langle \text{vac} | \hat{T}^{00} |\text{vac}\rangle = -\frac{1}{2} \sum_{\lambda} E_\lambda |\psi_\lambda^\dagger(r) \psi_\lambda(r)|. \]
Self-adjointness of the Dirac operator

Defining a scalar product as \((\tilde{\chi}, \chi) = \int d^3r \ \tilde{\chi}^\dagger \chi,\)

we get, using integration by parts,

\[
(\tilde{\chi}, H_{(1/2)} \chi) = (H_{(1/2)}^\dagger \tilde{\chi}, \chi) - i \int d\sigma \cdot J[\tilde{\chi}, \chi],
\]

where

\[
H_{(1/2)} = H_{(1/2)}^\dagger = -i \alpha \cdot \nabla + \beta m
\]

is the formal expression for the Dirac hamiltonian operator and

\[
J[\tilde{\chi}, \chi] = \tilde{\chi}^\dagger \alpha \chi.
\]

Operator \(H_{(1/2)}\) is Hermitian,

\[
(\tilde{\chi}, H_{(1/2)} \chi) = (H_{(1/2)}^\dagger \tilde{\chi}, \chi),
\]

if

\[
\int_{\partial D} d\sigma \cdot J[\tilde{\chi}, \chi] = 0 \quad \text{or} \quad n \cdot J[\tilde{\chi}, \chi]|_{r \in \partial D} = 0.
\]
To fulfill the latter condition, we impose the same boundary condition for $\chi$ and $\tilde{\chi}$ in the form

$$\chi|_{r \in \partial D} = K \chi|_{r \in \partial D}, \quad \tilde{\chi}|_{r \in \partial D} = K \tilde{\chi}|_{r \in \partial D},$$

where $K$ is a matrix (element of the Clifford algebra) which is determined by two conditions:

$$K^2 = I$$

and

$$K^\dagger (n \cdot \alpha) K = -n \cdot \alpha.$$

It should be noted that, in addition, the following combination of $\chi$ and $\tilde{\chi}$ is also vanishing at the boundary:

$$\tilde{\chi}^\dagger (n \cdot \alpha) K \chi|_{r \in \partial D} = \tilde{\chi}^\dagger K^\dagger (n \cdot \alpha) \chi|_{r \in \partial D} = 0.$$
Using the standard representation for the Dirac matrices,

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$$

($\sigma^1, \sigma^2$ and $\sigma^3$ are the Pauli matrices), one can get

$$K = \begin{pmatrix} 0 & \varrho^{-1} \\ \varrho & 0 \end{pmatrix},$$

where condition

$$(n \cdot \sigma) \varrho = -\varrho^\dagger (n \cdot \sigma)$$

defines $\varrho$ as a rank-2 matrix depending on four arbitrary parameters. An explicit form for matrix $K$ is

$$K = \left(1 + u^2 - v^2 - t^2\right)\beta + \left(1 - u^2 + v^2 + t^2\right)I \frac{un \cdot \alpha + v \beta \gamma^5 - it \cdot \alpha}{2i(u^2 - v^2 - t^2)},$$

where $\gamma^5 = i\alpha^1\alpha^2\alpha^3$, and $t = (t^1, t^2)$ is a two-dimensional vector which is tangential to the boundary.
Using parametrization

\[ u = \cosh \tilde{\vartheta} \cosh \vartheta, \quad v = \cosh \tilde{\vartheta} \sinh \vartheta \cos \theta, \]

\[ t^1 = \cosh \tilde{\vartheta} \sinh \vartheta \sin \theta \cos \phi, \quad t^2 = \cosh \tilde{\vartheta} \sinh \vartheta \sin \theta \sin \phi, \]

\[-\infty < \vartheta < \infty, \quad 0 \leq \tilde{\vartheta} < \infty, \quad 0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi \] (15)

in the case of \( u^2 - v^2 - t^2 \geq 1 \), one gets

\[ K = \frac{\beta (1 + \cosh^2 \tilde{\vartheta}) - I \sinh^2 \tilde{\vartheta}}{2i \cosh \tilde{\vartheta}} \]

\[ \times [n \cdot \alpha \cosh \vartheta + \beta \gamma^5 \sinh \vartheta \cos \theta - i(\alpha^1 \cos \phi + \alpha^2 \sin \phi) \sinh \vartheta \sin \theta] \]

and

\[ (I - K)\chi|_{r \in \partial D} = (I - K)\tilde{\chi}|_{r \in \partial D} = 0 \]

is the most general extension of the standard MIT bag boundary condition (K. Johnson, 1975); the latter is obtainable at \( \vartheta = \tilde{\vartheta} = \theta = \phi = 0 \).
In the case of $\partial D : \partial D^+ \oplus \partial D^-$, there are 8 self-adjoint extension parameters: $\vartheta_+, \tilde{\vartheta}_+,$ $\theta_+,$ and $\phi_+$ corresponding to $\partial D^+$ and $\vartheta_-, \tilde{\vartheta}_-, \theta_-$, and $\phi_-$ corresponding to $\partial D^-$. However, if some symmetry is present, then the number of self-adjoint extension parameters is diminished. For instance, if the boundary consists of two parallel planes, then the cases differing by the values of $\phi_+$ or $\phi_-$ are physically indistinguishable, since they are related by a rotation around a normal to the boundary. To avoid this unphysical degeneracy, one has to fix

$$\theta_+ = \theta_- = 0,$$

and there remains 4 self-adjoint extension parameters: $\vartheta_+, \tilde{\vartheta}_+, \vartheta_-$, and $\tilde{\vartheta}_-$. Operator $H_{(1/2)}$ acting on functions which are defined in the region bounded by two parallel planes is self-adjoint, if the following condition holds:

$$[ I - \frac{\beta (\cosh^2 \vartheta_\pm + 1) - I \sinh^2 \vartheta_\pm}{2i \cosh \tilde{\vartheta}_\pm} (\pm \alpha^2 \cosh \vartheta_\pm + \beta \gamma^5 \sinh \vartheta_\pm) ]$$

$$\times \chi|_{z = \pm a/2} = 0.$$
In the spin-1/2 case, any immediate physical motivation to diminish the number of the self-adjoint extension parameters seems to be lacking. In this situation one can be guided by such arguments as simplicity and unambiguity of the determination of the spectrum of $k_l - z$-component of the wave number vector. In particular, the condition that this spectrum be independent of the values of other components of the wave number vector yields restriction

$$\vartheta_+ = \vartheta_- = \vartheta, \quad \tilde{\vartheta}_+ = \tilde{\vartheta}_- = 0,$$

with resulting boundary condition

$$(I \pm i \beta \alpha^z \cosh \vartheta + i \gamma^5 \sinh \vartheta) \chi|_{z = \pm a/2} = 0.$$ 

The standard MIT bag boundary condition corresponds to $\vartheta = 0$:

$$(I \pm i \beta \alpha^z) \chi|_{z = \pm a/2} = 0.$$
Vacuum energy per unit area of the boundary surface

**scalar case:**

\[
\frac{E}{S} = \int_{-a/2}^{a/2} dz \varepsilon = \sum_{\lambda} \omega_{\lambda},
\]

where

\[
\sin \left[ \frac{1}{2} (k_l a + \rho) \right] = 0 \quad (\infty \leq k_l < \infty), \quad \rho \neq \pi/2, \ 3\pi/2
\]

or

\[
\cos (k_l a) = 0 \quad (0 \leq k_l < \infty), \quad \rho = \pi/2, \ 3\pi/2.
\]

**spinor case:**

\[
\frac{E}{S} = \int_{-a/2}^{a/2} dz \varepsilon = -\frac{1}{2} \sum_{\lambda} |E_\lambda|,
\]

where

\[
\sin \left[ k_l a + \arctan \left( \frac{\hbar k_l \cosh \vartheta}{mc} \right) \right] = 0 \quad (0 \leq k_l < \infty).
\]
Background: uniform magnetic field orthogonal to the boundary

\[ \mathbf{B} = (0, 0, B), \quad \mathbf{A} = (-yB, 0, 0) \]

\[ \partial D : \quad \partial D^{(+)} \oplus \partial D^{(-)} \]

\[ \mathbf{r} = (x, y, z) \quad \partial D^{(+)} : \quad z = a/2; \quad \partial D^{(-)} : \quad z = -a/2 \]

\[ \nabla \hat{\psi} = (\partial - ie\mathbf{A})\hat{\psi}, \quad \nabla \hat{\psi}^\dagger = (\partial + ie\mathbf{A})\hat{\psi}^\dagger, \quad \mathbf{B} = \partial \times \mathbf{A} \]

One-particle energy spectrum (Landau levels):

\[ \omega_{snl} = \sqrt{|eB|(2n + 1 - 2s) + k^2_f + m^2}, \]

\[ s = 0, 1/2, \quad n = 0, 1, 2, \ldots, \]
Vacuum energy per unit area of the boundary surface

\[
\frac{E_{(s)}}{S} = \frac{|eB|}{2\pi} (1 - 4s) \sum_{l} \sum_{n=0}^{\infty} (1 + 2s - 2s\delta_{n0})\omega_{snl}
\]

Abel-Plana summation formula and its generalizations, see


\[
\frac{E_{(s)}}{S} = a\varepsilon_{(s)}^\infty + \Omega_{(s)}(a) + \tilde{\Omega}_{(s)},
\]

where

\[
\varepsilon_{(s)}^\infty = \frac{|eB|}{(2\pi)^2} (1 - 4s) \int_{-\infty}^{\infty} dk \sum_{n=0}^{\infty} (1 + 2s - 2s\delta_{n0})\omega_{snk}.
\]
Regularization & renormalization: \( \varepsilon^\infty(s) \rightarrow \varepsilon^{\infty}_{\text{ren}}(s) \)

\[
\varepsilon^{\infty}_{\text{ren}} = \frac{e^2 B^2}{(4\pi)^2} \int_0^\infty \frac{d\eta}{\eta} \exp \left( -\frac{m^2 \eta}{|eB|} \right) \left[ \frac{4s \cosh \eta - 1 + 2s}{\eta \sinh \eta} \right] \\
+(1 - 6s) \frac{1}{\eta^2} - \frac{1}{6} (1 + 6s)
\]


Regularization & renormalization: \( \frac{E(s)}{S} \rightarrow \frac{E_{\text{ren}}(s)}{S} \)

\[
\frac{E_{\text{ren}}}{S} = a \varepsilon^{\infty}_{\text{ren}}(s) + \Omega(s)(a) + \tilde{\Omega}(s)
\]
Casimir force (or pressure)

\[ F(s) \equiv -\frac{\partial}{\partial a} \frac{E(s)_{\text{ren}}}{S} = -\varepsilon_{(s)\text{ren}} + \Delta(s)(a), \]

where

\[ \Delta(s)(a) \equiv -\frac{\partial}{\partial a} \Omega(s)(a) \]

\[ = -\frac{|eB|}{\pi^2} \sum_{n=0}^{\infty} (1 + 2s - 2s\delta_{n0}) \int_{M_{sn}}^{\infty} d\kappa \gamma(s)(\kappa) \kappa^{2-4s}(\kappa^2 - M_{sn}^2)^{2s-1/2}, \]

\[ M_{sn} = \sqrt{|eB|(2n + 1 - 2s) + m^2}, \]

\[ \gamma(0)(\kappa) = \frac{1}{2} \frac{\cos \rho - e^{-\kappa a}}{\cosh(\kappa a) - \cos \rho}, \]

\[ \gamma(1/2)(\kappa) = \frac{\left(2\kappa a - 1\right) \left(\kappa^2 \cosh^2 \vartheta - m^2\right) - 2\kappa m \cosh \vartheta}{\left(\kappa \cosh \vartheta + m\right) e^{2\kappa a} + \kappa \cosh \vartheta - m} e^{2\kappa a} \]

\[ \frac{\left(\kappa \cosh \vartheta - m\right)^2}{\left(\kappa \cosh \vartheta + m\right) e^{2\kappa a} + \kappa \cosh \vartheta - m} \]
\[ F(s) = -\varepsilon^{\infty}_{(s)\text{ren}} + \Delta(s)(a), \]

\(-\varepsilon^{\infty}_{(s)\text{ren}}\) is positive

In the case of a weak magnetic field, \(|B| \ll m^2|e|^{-1}\), one has

\[-\varepsilon^{\infty}_{(s)\text{ren}} = \frac{1}{360\pi^2} \left[ 1 - \frac{9}{8} \left( \frac{1}{2} - s \right) \right] \left( \frac{eB}{m} \right)^4.\]

Note that the critical value is the lowest one, \(B_{\text{crit}} = m^2|e|^{-1} = 4.41 \times 10^{13}\) Gauss, for the case of quantized electron-positron matter.

In the case of a strong magnetic field, \(|B| \gg m^2|e|^{-1}\), one has

\[-\varepsilon^{\infty}_{(s)\text{ren}} = \frac{1}{24\pi^2} \left[ 1 - \frac{3}{2} \left( \frac{1}{2} - s \right) \right] e^2 B^2 \ln \frac{2|eB|}{m^2}.\]
\( \Delta_{(1/2)}(a) \) at \( |B| \ll m^2|e|^{-1} \) takes the forms in the limits of large and small distances between the plates

\[
\Delta_{(1/2)}(a) = \begin{cases} 
- \frac{3}{16\pi^{3/2}} \frac{m^{3/2}}{a^{5/2}} e^{-2ma[1 + O(\frac{1}{ma})]} , & \vartheta = 0 \\
- \frac{\tanh^2(\vartheta/2)}{2\pi^{3/2}} \frac{m^{5/2}}{a^{3/2}} e^{-2ma[1 + O(\frac{1}{ma})]} , & \vartheta \neq 0 
\end{cases}
\]

\(|eB| \ll m^2, \quad ma \gg 1\)

and

\[
\Delta_{(1/2)}(a) = -\frac{7}{8} \frac{\pi^2}{120} \frac{1}{a^4}, \quad |eB| \ll m^2, \quad ma \ll 1.
\]
\( \Delta_{(1/2)}(a) \) at \( |B| \gg m^2 |e|^{-1} \) takes the forms in the limits of large and small distances between the plates

\[
\Delta_{(1/2)}(a) = \begin{cases} 
- \frac{|eB|}{16 \pi^{3/2}} \frac{m^{1/2}}{a^{3/2}} e^{-2ma} [1 + O(\frac{1}{ma})], & \vartheta = 0 \\
- \frac{|eB| \tanh^2(\vartheta/2)}{2 \pi^{3/2}} \frac{m^{3/2}}{a^{1/2}} e^{-2ma} [1 + O(\frac{1}{ma})], & \vartheta \neq 0 \end{cases}
\]

\[
\sqrt{|eB|} a \gg ma \gg 1
\]

and

\[
\Delta_{(1/2)}(a) = - \frac{|eB|}{48 a^2}, \quad ma \ll 1, \, \sqrt{|eB|} a \gg 1.
\]

\[
m^{-1} = 3.86 \times 10^{-13} \text{ m}, \quad a > 10^{-8} \text{ m}
\]


The pressure from the vacuum of confined charged massive matter in the background of a magnetic field orthogonal to plates is positive, being independent of the choice of a boundary condition, as well as of the distance between the plates.
Thank you for your attention!