Characterization of compact and self-adjoint operators, and study of positive operators on a Banach space over a non-Archimedean field

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- **Characterization of compact and self-adjoint operators on Free Banach spaces of countable type over the complex Levi-Civita field,** ***Journal of Mathematical Physics, Volume 54 # 2, 2013.***

- **Inner product on B*-algebras of operators on a free Banach space over the Levi-Civita field,** ***Indagationes Mathematicae*****, Volume 26 # 1, 2015.**

- **Positive operators on a Banach space over the Levi-Civita field,** ***in preparation.***

Contents

1. The Levi-Civita Fields $\mathcal{R}$ and $\mathcal{C}$

- **Let** $\mathcal{R} = \{f : \mathbb{Q} \to \mathbb{R} | \{q \in \mathbb{Q} | f(q) \neq 0\}$ **is left-finite**$\}$**.**

- **Notation: An element of** $\mathcal{R}$ **is denoted by** x **and its function value at** $q \in \mathbb{Q}$ **by** $x[q]$**.**

- **Supp**$(x) = \{q \in \mathbb{Q} |\ x[q] \neq 0\}$**.**

- **For** $x \in \mathcal{R}$**, define**
$$\lambda(x) = \begin{cases} \textbf{min(supp(}x\textbf{))} & \textbf{if } x \neq 0 \\ \infty & \textbf{if } x = 0 \end{cases}.$$

- **Arithmetic on** $\mathcal{R}$**: Let** $x, y \in \mathcal{R}$**. We define** $x + y$ **and** $x \cdot y$ **as follows. For** $q \in \mathbb{Q}$**, let**
$$\begin{aligned} (x+y)[q] &= x[q] + y[q] \\ (x \cdot y)[q] &= \sum_{q_1+q_2=q} x[q_1] \cdot y[q_2]. \end{aligned}$$
Then, $x + y \in \mathcal{R}$ **and** $x \cdot y \in \mathcal{R}$**.**

Result: $(\mathcal{R}, +, \cdot)$ **is a field. [Levi-Civita, 1892]**

Definition: $\mathcal{C} := \mathcal{R} + i\mathcal{R}$**. Then** $(\mathcal{C}, +, \cdot)$ **is also a field.**

Order in $\mathcal{R}$

- **Define the relation $\leq$ on $\mathcal{R} \times \mathcal{R}$ as follows:**

$x \leq y$ **if** $x = y$ **or** $(x \neq y$ **and** $(x - y)[\lambda(x - y)] < 0)$.

- $(\mathcal{R}, +, \cdot, \leq)$ **is an ordered field.**

- $\mathcal{R}$ **is real closed.**

$$\Downarrow$$

$\mathcal{C}$ **is algebraically closed.**

- **The map $E : \mathbb{R} \to \mathcal{R}$, given by**

$$E(x)[q] = \begin{cases} x & \textbf{if } q = 0 \\ 0 & \textbf{else} \end{cases},$$

is an order preserving embedding.

- **There are infinitely small and infinitely large elements in $\mathcal{R}$: The number d, given by**

$$d[q] = \begin{cases} 1 & \textbf{if } q = 1 \\ 0 & \textbf{else} \end{cases},$$

is infinitely small; while d^{-1} is infinitely large.

For $x \in \mathcal{R}$, define

$$|x|_0 = \begin{cases} x & \textbf{if } x \geq 0 \\ -x & \textbf{if } x < 0 \end{cases};$$

$$|x| = \begin{cases} e^{-\lambda(x)} & \textbf{if } x \neq 0 \\ 0 & \textbf{if } x = 0 \end{cases}.$$

For $z = x + iy \in \mathcal{C}$, define

$$|z|_0 = \sqrt{|x|_0^2 + |y|_0^2} = \sqrt{x^2 + y^2};$$

$$\begin{aligned} |z| &= \begin{cases} e^{-\lambda(z)} & \textbf{if } z \neq 0 \\ 0 & \textbf{if } z = 0 \end{cases} \\ &= \max\{|x|, |y|\} \textbf{ since } \lambda(z) = \min\{\lambda(x), \lambda(y)\}. \end{aligned}$$

Note that $|\cdot|_o$ and $|\cdot|$ induce the same topology τ_v on $\mathcal{R}$ (or $\mathcal{C}$). Moreover, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ in $\mathcal{R}$.

Properties of $(\mathcal{R}, \tau_v)$

- $(\mathcal{R}, \tau_v)$ **is a disconnected topological space.**

- $(\mathcal{R}, \tau_v)$ **is Hausdorff.**

- **There are no countable bases.**

- **The topology induced to $\mathbb{R}$ is the discrete topology.**

- $(\mathcal{R}, \tau_v)$ **is not locally compact.**

- τ_v **is zero-dimensional (i.e. it has a base consisting of clopen sets).**

- τ_v **is not a vector topology.**

- **For all $x \in \mathcal{R}$ (or $\mathcal{C}$): $x = \sum_{n=1}^{\infty} x[q_n] \cdot d^{q_n}$.**

Uniqueness of $\mathcal{R}$ and $\mathcal{C}$

- **$\mathcal{R}$ is the smallest complete and real closed non-Archimedean field extension of $\mathbb{R}$.**
  - **It is small enough so that the $\mathcal{R}$-numbers can be implemented on a computer, thus allowing for computational applications.**

- **$\mathcal{C}$ is the smallest complete and algebraically closed non-Archimedean field extension of $\mathbb{R}$ (or $\mathbb{C}$).**

2. Normal Projections, Compact and Self-adjoint Operators on $c_0(\mathcal{C})$

$$
\begin{aligned}
c_0(\mathcal{C}) &= \left\{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{C}; \lim_{n\to\infty} x_n = 0\right\} \equiv c_0;\\
c_0(\mathcal{R}) &= \left\{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{R}; \lim_{n\to\infty} x_n = 0\right\};\\
\mathcal{L}(c_0) &= \{T : c_0 \to c_0 : T \textbf{ linear \& continuous}\}.
\end{aligned}
$$

We consider the following form:

$$\langle\cdot,\cdot\rangle : c_0 \times c_0 \to \mathcal{C};\ \langle z, w\rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}.$$

This is well-defined; and it satisfies:

(1) $\langle z, z\rangle \geq 0$ **and** $\langle z, z\rangle = 0$ **if and only if** $z = 0$**;**

(2) $\left\langle az^1 + bz^2, w\right\rangle = a\left\langle z^1, w\right\rangle + b\left\langle z^2, w\right\rangle$ **for** $a, b \in \mathcal{C}$ **and** $z^1, z^2, w \in c_0$;

(3) $\langle z, w\rangle = \overline{\langle w, z\rangle}$ **for** $z, w \in c_0$**;**

(4) $|\langle z, w\rangle|^2 \leq |\langle z, z\rangle|\,|\langle w, w\rangle|$ **(the Cauchy-Schwarz inequality).**

Let $\|z\| := \sqrt{|\langle z, z\rangle|}$**. Then** $\|\cdot\|$ **is a non-Archimedean norm on** c_0**. Moreover,** $\|\cdot\| = \|\cdot\|_\infty$**.**

<u>Notation</u>: If M is a subspace of c_0, then M^p will denote the subspace of all $y \in c_0$ such that $\langle y, x \rangle = 0$, for all $x \in M$.

<u>Definition</u>: A sequence (z^n) of non-null vectors of c_0 has the Riemann-Lebesgue Property (RLP) if for all $z \in c_0$,

$$\lim_{n \to \infty} \langle z^n, z \rangle = 0.$$

<u>Note</u>: Any basis of c_0 has the (RLP) property.

<u>Theorem</u>: Let M be an infinite dimensional closed subspace of c_0. Then, the following statements are equivalent:

(1) M has a normal complement. That is $c_0 = M \oplus M^p$.

(2) M has an orthonormal base with the Riemann-Lebesgue Property.

(3) There exists a normal projection P such that $N(P) = M$.

Any continuous linear operator $u \in \mathcal{L}(c_0)$ can be identified with a matrix of the form

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots \\ \vdots & & & \ddots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots \\ \vdots & & & & & \ddots \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ 0 & 0 & 0 & & 0 & \end{pmatrix}$$

where

(1) $\lim_{i\to\infty} \alpha_{ij} = 0$, **for any** $j \in \mathbb{N}$,

(2) $\sup_{i,j\in\mathbb{N}} |\alpha_{ij}| < \infty$,

(3) $\|u\| = \sup_{i,j\in\mathbb{N}} |\alpha_{ij}| = \sup_{n\in\mathbb{N}} \|ue_n\|$.

<u>Definition</u>: An operator $v : c_0 \to c_0$ is said to be an adjoint of a given operator $u \in \mathcal{L}(c_0)$ if $\langle u(x), y\rangle = \langle x, v(y)\rangle$, for all $x, y \in c_0$. In that case, we will say that u admits an adjoint v. We will also say that u is self-adjoint if $v = u$.

Lemma: Let $u \in \mathcal{L}(c_0)$ with associated matrix $\{\alpha_{i,j}\}_{i,j\geq 1}$. Then, u admits an adjoint operator v if and only if $\lim_{j\to\infty} |\alpha_{ij}| = 0$, for each $i \in \mathbb{N}$. So

$$[u] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots & \to 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots & \to 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots & \to 0 \\ \vdots & & & \ddots & & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i\mathbf{3}} & \cdots & \alpha_{ij} & \cdots & \to 0 \\ \vdots & & & & & \ddots & \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\ 0 & 0 & 0 & & 0 & & \end{pmatrix}.$$

In the classical Hilbert space theory, any continuous linear operator admits an adjoint. This is not true in the non-Archimedean case. For example, the operator $u \in \mathcal{L}(c_0)$ given by the matrix:

$$\begin{pmatrix} b & b^2 & b^3 & \cdots & b^j & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix},$$

with $1 < |b|$, doesn't admit an adjoint.

The following two theorems provide characterizations for normal projections.

<u>Theorem</u>: Let $P \in \mathcal{L}(c_0)$. Then P is a normal projection if and only if P is self-adjoint and $P^2 = P$.

<u>Theorem</u>: If $P : c_0 \to c_0$ is a normal projection with $R(P) = \overline{[\{y_1, y_2, \cdots\}]}$, where $\{y_1, y_2, \cdots\}$ is an orthonormal finite subset of c_0 or an orthonormal sequence with the Riemann-Lebesgue Property, then $Px = \sum_{i=1}^{\infty} \frac{\langle x, y_i \rangle}{\langle y_i, y_i \rangle} y_i$.

Since $\mathcal{C}$ is not locally compact, convex compact sets of c_0 are trivial.

__Definition__: **A subset C of c_0 is called compactoid if for every $\epsilon > 0$ there exists a finite subset $S \subset c_0$ such that $C \subset B_{c_0}(\mathbf{0}, \epsilon) + \overline{co}(S)$, where $B_{c_0}(\mathbf{0}, \epsilon) = \{x \in c_0 : \|x\| \leq \epsilon\}$ and $\overline{co}(S)$ is the absolutely (closed) convex hull of S.**

__Definition__: **A linear operator $T : c_0 \to c_0$ is said to be compact if $T(B_{c_0})$ is compactoid, where $B_{c_0} = \{x \in c_0 : \|x\| \leq 1\}$ is the unit ball of c_0.**

__Theorem__: **$T \in \mathcal{L}(c_0)$ is compact if and only if, for each $\epsilon > 0$, there exists a linear operator of finite-dimensional range S such that $\|T - S\| \leq \epsilon$.**

The following theorem provides a way to construct compact and self-adjoint operators starting from an orthonormal sequence.

__Theorem__: Let $\{y_1, y_2, \cdots\}$ be an orthonormal sequence in c_0. Then, for any $\lambda = (\lambda_n)$ in $c_0(\mathcal{R})$, the map $T : c_0 \to c_0$ defined by

$$T(\cdot) = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n \equiv \sum_{n=1}^{\infty} \lambda_n P_n(\cdot)$$

is a compact and self-adjoint operator.

The converse is also true, as the following theorem shows.

__Theorem__: If the linear operator $T : c_0 \to c_0$ is compact and self-adjoint, then there exist $\lambda = (\lambda_n) \in c_0(\mathcal{R})$ and an orthonormal sequence $\{y_n\}$ in c_0 such that

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

Let $\mathcal{A}_0 \equiv \mathcal{A}_0(c_0) := \{T \in \mathcal{L}(c_0) : T$ **has an adjoint**$\}$**.**

- $\mathcal{A}_0$ **is a non-commutative Banach algebra with unity.**

- $\mathcal{A}_0$ **contains normal projections.**

- $T \in \mathcal{A}_0$ **if and only if its associated matrix has the form:**

$$[T] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & & \alpha_{1j} & \rightarrow & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & & \alpha_{2j} & \rightarrow & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & & \alpha_{3j} & \rightarrow & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & & \alpha_{ij} & \rightarrow & 0 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \cdots & \\ 0 & 0 & 0 & & 0 & \cdots & \end{bmatrix}$$

- $\mathcal{A}_0$ **can be rewritten as**

$$\mathcal{A}_0 = \left\{ T \in \mathcal{L}(c_0) : \forall y \in c_0, \ \lim_{j\to\infty} \langle Te_j, y\rangle = 0 \right\}.$$

- **For** $T \in \mathcal{A}_0$**,** $T^* \in \mathcal{A}_0$ **and** $(T^*)^* = T^{**} = T$**. Therefore, the map** $^* : \mathcal{A}_0 \to \mathcal{A}_0; \ T \to T^*$, **is an involution on** $\mathcal{A}_0$.

- **Altogether, we say** $\mathcal{A}_0$ **is a non-Archimedean** B^*-**algebra.**

Let $\mathcal{A}_1 = \left\{ T \in \mathcal{L}(c_0) : \lim_{n\to\infty} Te_n = 0 \right\}$**.**

- **From** $|\langle Te_n, y\rangle| \leq \|Te_n\| \|y\|$ **, we have that** $\mathcal{A}_1 \subset \mathcal{A}_0$**. But** $\mathcal{A}_1 \neq \mathcal{A}_0$ **since** $Id \notin \mathcal{A}_1$.

- $\mathcal{A}_1$ **is a closed subalgebra of** $\mathcal{A}_0$.

- $T \in \mathcal{A}_1 \Leftrightarrow T$ **is compact and** $T \in \mathcal{A}_0$**.**

Let $\mathcal{A}_2 = \{T \in \mathcal{A}_1 : T = T^*\}$.

- **Note that the operator**

$$S(\cdot) = \sum_{i=1}^{\infty} a_i \langle \cdot, e_i \rangle e_{i+1},$$

  where $a = (a_i) \in c_0$, is in $\mathcal{A}_1$, but it is not self-adjoint; therefore $\mathcal{A}_2$ is a proper subset of $\mathcal{A}_1$.

- **$\mathcal{A}_2$ is a closed subset of $\mathcal{A}_1$.**

- **$T \in \mathcal{A}_2$ if and only if there exist an element $(\lambda_n) \in c_0(\mathcal{R})$ and an orthonormal sequence $\{y_n\}_{n\in\mathbb{N}}$ in c_0 such that**

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

  - **$\|T\| = \|(\lambda_n)\|$.**

  - **$\mathcal{A}_2 \cong c_0(\mathcal{R})$.**

Inner Product in $\mathcal{A}_1$

Since $\lim_{n\to\infty} Se_n = 0$ **and** $\lim_{n\to\infty} Te_n = 0$ **for** $S, T \in \mathcal{A}_1$**, the mapping**

$$\langle \cdot, \cdot \rangle : \mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{C}; \quad (S, T) \to \langle S, T \rangle = \sum_{i=1}^{\infty} \langle Se_i, Te_i \rangle \,,$$

is well-defined, linear in the first variable and linear conjugate in the second variable.

- $\langle S, T \rangle = \overline{\langle T, S \rangle}$ **for all** $S, T \in \mathcal{A}_1$.
- $\langle S, S \rangle \geq 0$ **and** $\langle S, S \rangle = 0 \Leftrightarrow S = 0$**.**
- $\sqrt{|\langle S, S \rangle|} = \|S\|$ **for all** $S \in \mathcal{A}_1$**.**
- $|\langle S, T \rangle| \leq \|S\| \|T\|$ **for all** $S, T \in \mathcal{A}_1$**.**
- c_0 **is isometrically isomorphic to a closed subspace** $\mathcal{S}$ **of** $\mathcal{A}_1$. **Moreover, the restriction of the inner product in** $\mathcal{A}_1$ **to** $\mathcal{S}$ **coincides with the inner product defined in** c_0.

4. Positive operators

Definition: For $T \in \mathcal{A}_1$, we say that T is positive and write $T \geq 0$ if $\langle Tx, x \rangle \geq 0$ for all $x \in c_0(\mathcal{C})$.

Proposition: Let $S, T \geq 0$ in $\mathcal{A}_1$ and $\alpha \geq 0$ in $\mathcal{R}$ be given. Then

- **$\alpha S + T \geq 0$.**
- **T is self-adjoint; that is $T \in \mathcal{A}_2$.**
- **For all $x, y \in c_0$, we have that**

$$|\langle Tx, y \rangle|^2 \leq |\langle Tx, x \rangle| \, |\langle Ty, y \rangle| \, .$$

Proposition: Let $T \in \mathcal{A}_1$. Then both TT^* and T^*T are positive.

Theorem: For $T \in \mathcal{A}_1$, the following are equivalent:

(1) $T \geq 0$**.**
(2) T **is self-adjoint; and all of its eigenvalues are in** $\mathcal{R}$ **and non-negative.**
(3) There exists $S \geq 0$ **in** $\mathcal{A}_1$ **such that** $T = S^2$**.**
(4) There exists $S \in \mathcal{A}_1$ **such that** $T = S^*S$**.**

Proof: $(1) \Rightarrow (2)$**: Assume** $T \geq 0$**. Then** T **is self-adjoint. Now let** λ **be an eigenvalue of** T **and let** $v \in c_0(\mathcal{C})$ **be a corresponding eigenvector. Then**

$$0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle.$$

Since $\langle v, v \rangle > 0$**, it follows that** $\lambda \in \mathcal{R}$ **and** $\lambda \geq 0$**.**

$(2) \Rightarrow (3)$**: Assume (2) is true. Since** T **is compact and self-adjoint, there exist** $(\lambda_n) \in c_0(\mathcal{R})$ **and an orthonormal sequence** $\{y_n\}$ **in** $c_0(\mathcal{C})$ **such that**

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

For each $m \in \mathbb{N}$**, we have that**

$$Ty_m = \sum_{n=1}^{\infty} \lambda_n \frac{\langle y_m, y_n \rangle}{\langle y_n, y_n \rangle} y_n = \lambda_m y_m.$$

Thus, λ_n is an eigenvalue of T for each n and hence $\lambda_n \in \mathcal{R}$ and $\lambda_n \geq 0$ for all $n \in \mathbb{N}$. Let $S : c_0(\mathcal{C}) \to c_0(\mathcal{C})$ be given by

$$S = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

Then $S \in \mathcal{A}_1$. For all $x \in c_0(\mathcal{C})$, we have that

$$\begin{aligned} \langle Sx, x \rangle &= \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} \langle y_n, x \rangle \\ &= \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle x, y_n \rangle \overline{\langle x, y_n \rangle}}{\langle y_n, y_n \rangle} \geq 0. \end{aligned}$$

Hence $S \geq 0$. Also, for all $x \in c_0(\mathcal{C})$:

$$\begin{aligned} S^2 x = S(Sx) &= S\left(\sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} y_n \right) \\ &= \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} S\left(y_n \right) \\ &= \sum_{n=1}^{\infty} \lambda_n \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} y_n = Tx. \end{aligned}$$

Hence $S^2 = T$.

$(3) \Rightarrow (4)$**: Assume there exists $S \geq 0$ in $\mathcal{A}_1$ such that $T = S^2$. Then S is self-adjoint. Thus, $S = S^*$ and hence $T = S^2 = SS = S^* S$.**

$(4) \Rightarrow (1)$**: Follows from previous proposition.**

Remark: **Let T and S be as in the previous theorem. Then**

- **S is unique. We say S is the positive square root of T and write $S = \sqrt{T}$.**

- **Moreover,**

$$\|S\| = \|(\sqrt{\lambda_n})\| = \max_{n\in\mathbb{N}} \left\{|\sqrt{\lambda_n}|\right\} = \max_{n\in\mathbb{N}} \left\{|\lambda_n|^{1/2}\right\}$$
$$= \left[\max_{n\in\mathbb{N}} \{|\lambda_n|\}\right]^{1/2} = \|(\lambda_n)\|^{1/2} = \|T\|^{1/2}.$$

Proposition: Let $T \geq 0$ in $\mathcal{A}_1$, let $S = \sqrt{T}$, and let $R \in \mathcal{A}_1$ be given. Then $TR = RT \Leftrightarrow SR = RS$.

Proposition: Let $T \geq 0$ in $\mathcal{A}_1$ and $x \in c_0(\mathcal{C})$ be given. Then $\langle Tx, x \rangle = 0$ if and only if $Tx = 0$.

Proof: If $Tx = 0$ then $\langle Tx, x \rangle = 0$ by definition of the inner product. Now assume $\langle Tx, x \rangle = 0$. Then, since $T \geq 0$, there exists $S \in \mathcal{A}_1$ such that $T = S^*S$. Thus, $\langle S^*Sx, x \rangle = 0$, and hence $\langle Sx, Sx \rangle = 0$, from which we get $Sx = 0$. It follows that $Tx = S^*Sx = S^*0 = 0$.

Corollary: Let $T \geq 0$ in $\mathcal{A}_1$. Then $\langle Tx, x \rangle = 0$ for all $x \in c_0(\mathcal{C})$ if and only if $T = 0$.

Proposition: Let $S, T \in \mathcal{A}_1$ be positive. Then $ST \geq 0 \Leftrightarrow ST = TS$.

Proof: $(\Rightarrow)$: Easy.

$(\Leftarrow)$: Assume $ST = TS$. Let $N = \sqrt{T}$. Applying a previous proposition, we have that $NS = SN$. Now let $x \in c_0(\mathcal{C})$ be given. Then

$$\begin{aligned}\langle STx, x \rangle &= \langle S(NN)x, x \rangle = \langle (SN)Nx, x \rangle \\ &= \langle (NS)Nx, x \rangle = \langle N(SN)x, x \rangle \\ &= \langle SNx, N^*x \rangle = \langle SNx, Nx \rangle \geq 0.\end{aligned}$$

Proposition: Let $T \in \mathcal{A}_2$ be given. Then there exist unique positive operators A and B such that $T = A - B$ and $AB = BA = 0$.

Proof: Since T is compact and self-adjoint, there exist $(\lambda_n) \in c_0(\mathcal{R})$ and an orthonormal sequence $\{y_n\}$ in $c_0(\mathcal{C})$ such that $T = \sum\limits_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$. Thus,

$$T = \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n + \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$$

$$= \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n - \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n) \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n = A - B.$$

For all $x \in c_0(\mathcal{C})$, we have that

$$ABx = \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n) \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} A(y_n)$$

$$= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\infty} (-\lambda_n) \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} \left(\sum_{\substack{l=1 \\ \lambda_l > 0}}^{\infty} \lambda_l \frac{\langle y_n, y_l \rangle}{\langle y_l, y_l \rangle} y_l \right) = 0.$$

Hence $AB = 0$. It follows that $BA = 0$ too.

Uniqueness: Assume that $T = A_1 - B_1$ with A_1 and B_1 positive operators in $\mathcal{A}_2$ and $A_1 B_1 = B_1 A_1 = 0$. Write

$$A_1 = \sum_{l=1}^{\infty} \alpha_l \frac{\langle \cdot, x_l \rangle}{\langle x_l, x_l \rangle} x_l \text{ and } B_1 = \sum_{j=1}^{\infty} \beta_j \frac{\langle \cdot, z_j \rangle}{\langle z_j, z_j \rangle} z_j.$$

Fix $l_0 \in \mathbb{N}$. Then

$$\begin{aligned} T x_{l_0} &= (A_1 - B_1) x_{l_0} = A_1 x_{l_0} - B_1 x_{l_0} \\ &= \alpha_{l_0} x_{l_0} - B_1 A_1 \left(\frac{1}{\alpha_{l_0}} x_{l_0} \right) \\ &= \alpha_{l_0} x_{l_0}, \text{ since } B_1 A_1 = 0. \end{aligned}$$

Thus, α_{l_0} is an eigenvalue of T; and hence α_{l_0} is equal to some $\lambda_n > 0$. Similarly we show that, for each $j \in \mathbb{N}$, $-\beta_j$ is equal to some $\lambda_n < 0$. Hence

$$\{\alpha_l : l \in \mathbb{N}\} = \{\lambda_n : n \in \mathbb{N}, \lambda_n > 0\}$$

and

$$\{-\beta_j : j \in \mathbb{N}\} = \{\lambda_n : n \in \mathbb{N}, \lambda_n < 0\}.$$

It then follows that $A_1 = A$ and $B_1 = B$.

Proposition: Let T, A and B be as above. Then

(1) $\langle A, B\rangle = 0$**; and**

(2) $\|T\| = \max\{\|A\|, \|B\|\}$**.**

Proof:

(1) **First note that since A and B are positive operators, they are both self-adjoint. Thus,**

$$\langle A, B\rangle = \sum_{n=1}^{\infty} \langle Ae_n, Be_n\rangle = \sum_{n=1}^{\infty} \langle B^* Ae_n, e_n\rangle$$
$$= \sum_{n=1}^{\infty} \langle BAe_n, e_n\rangle = \sum_{n=1}^{\infty} \langle 0, e_n\rangle = 0.$$

(2) **Also** $\langle B, A\rangle = \overline{\langle A, B\rangle} = 0$**. Thus,**

$$\langle T, T\rangle = \langle A - B, A - B\rangle = \langle A, A\rangle + \langle B, B\rangle ;$$

and hence

$$|\langle T, T\rangle| = |\langle A, A\rangle + \langle B, B\rangle| = \max\{|\langle A, A\rangle|, |\langle B, B\rangle|\}$$

or, equivalently,

$$\|T\|^2 = \max\left\{\|A\|^2, \|B\|^2\right\} = (\max\{\|A\|, \|B\|\})^2 .$$

It follows that $\|T\| = \max\{\|A\|, \|B\|\}$**.**

5. Partial Order on $\mathcal{A}_2$

<u>Definition</u>: For $S, T \in \mathcal{A}_2$, we say that $S \geq T$ (or $T \leq S$) if $S - T \geq 0$.

<u>Theorem</u>: The relation $\geq$ defines a partial order on $\mathcal{A}_2$.

- **Reflexivity: For all $T \in \mathcal{A}_2$, $T \geq T$.**
- **Antisymmetry: Let $S, T \in \mathcal{A}_2$ be such that $S \geq T$ and $T \geq S$. Then $S - T \geq 0$ and $T - S \geq 0$. Thus, for all $x \in c_0(\mathcal{C})$:**

  $$\langle (S-T)x, x\rangle \geq 0 \text{ and } \langle (T-S)x, x\rangle \geq 0,$$

  from which we get

  $$\langle (S-T)x, x\rangle = 0$$

  for all $x \in c_0(\mathcal{C})$. It follows that $S - T = 0$ and hence $S = T$.
- **Transitivity: Let $R, S, T \in \mathcal{A}_2$ be such that $R \geq S$ and $S \geq T$. Then $R - S \geq 0$ and $S - T \geq 0$. It follows that**

  $$R - T = (R - S) + (S - T) \geq 0,$$

  and hence $R \geq T$.

Example: Let $S, T \in \mathcal{A}_2$ be the operators given by the matrix representations

$$[S] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } [T] = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$[S-T] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \ [T-S] = \begin{bmatrix} -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since both $S - T$ and $T - S$ have a negative eigenvalue (-1), it follows that neither is ≥ 0.

Proposition: If $S \geq T$ and $U \geq V$ in $\mathcal{A}_2$ then $S + U \geq T + V$. Moreover, if $S \geq T$ in $\mathcal{A}_2$ and $\alpha \geq 0$ in $\mathcal{R}$ then $\alpha S \geq \alpha T$.

Proof: Let $S, T, U, V \in \mathcal{A}_2$ be such that $S \geq T$ and $U \geq V$; and let $\alpha \geq 0$ in $\mathcal{R}$ be given. Then

$$(S + U) - (T + V) = (S - T) + (U - V) \geq 0$$

$$\alpha S - \alpha T = \alpha(S - T) \geq 0$$

For $R, S, T \in \mathcal{A}_2$: $R \geq 0$ and $S \geq T \not\Rightarrow SR \geq TR$.

<u>**Example**</u>**: Let R, S, T in $\mathcal{A}_2$ be given by**

$$[R] = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix};$$

$$[S] = \begin{bmatrix} 2 & -1 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}; [T] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $R \geq 0$ and

$$[S-T] = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \textbf{ so } S - T \geq 0$$

since, for all $x \in c_0(\mathcal{C})$, we have that

$$\begin{aligned} \langle (S-T)x, x \rangle &= \overline{x_1}(x_1 - x_2) + \overline{x_2}(x_2 - x_1) \\ &= |x_1|_o^2 - \overline{x_1}x_2 - \overline{x_2}x_1 + |x_2|_o^2 \\ &= |x_1|_o^2 - 2\mathcal{R}\left(\overline{x_1}x_2\right) + |x_2|_o^2 \\ &\geq |x_1|_o^2 - 2|x_1|_o|x_2|_o + |x_2|_o^2 \\ &= (|x_1|_o - |x_2|_o)^2 \geq 0, \end{aligned}$$

where, for $z = \alpha + i\beta \in \mathcal{C}$, $\mathcal{R}(z) = \alpha$ denotes the $\mathcal{R}$-part of the $\mathcal{C}$-number z.

However,

$$[SR] = \begin{bmatrix} 0 & -1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } [TR] = 0.$$

Thus

$$[SR - TR] = \begin{bmatrix} 0 & -1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore $SR - TR \not\geq 0$ since it is not self-adjoint; and hence $SR \not\geq TR$.

<u>Proposition</u>: Let $S, T \in \mathcal{A}_2$ be given. Then

$$S \geq T \Leftrightarrow \langle Sx, x\rangle \geq \langle Tx, x\rangle \text{ for all } x \in c_0(\mathcal{C}).$$

<u>Proposition</u>: Let $S \geq T$ in $\mathcal{A}_2$ and let $R \in \mathcal{A}_1$ be given. Then $R^*SR \geq R^*TR$.

<u>Proof</u>: Note that $R^*SR, R^*TR \in \mathcal{A}_2$. Let $x \in c_0(\mathcal{C})$ be given. Then

$$\begin{aligned} \langle (R^*SR - R^*TR)x, x\rangle &= \langle R^*(S-T)Rx, x\rangle \\ &= \langle (S-T)Rx, Rx\rangle \geq 0 \end{aligned}$$

since $S - T \geq 0$. Thus, $R^*SR - R^*TR \geq 0$, and hence $R^*SR \geq R^*TR$.