On the elliptic time in the Newton-Kepler adelic dynamic

George SHABAT

(IUM, MSU, RSUH, ITEP; Moscow)

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Plan of the talk

0. Physical models: transcendental vs. algebraic; $\overline{\mathbb{Q}}$ emerging
1. Two-body problem: the classical study over $\mathbb{R}$
2. The angular "time"
3. Two-body problem over an arbitrary field
4. Elliptic "time"
5. p-adic and adelic fantasies
6. Epilogue: Hitchin's ASD Einstein metrics, Painleve-VI and Poncelet theorem

0. Physical models: transcendental vs. algebraic

Algebro-geometric nature – often $/\overline{\mathbf{Q}}$ – of seemingly transcendent objects

Random examples (not for today's talk)
Voevodsky-Shabat,1987:

$$\dots \int \dots \int_{\mathrm{Met}(\mathbf{S}_g)} \longleftrightarrow \sum_{\Delta_g} \longleftrightarrow \sum_{\mathrm{BelPairs}_g(\overline{\mathbf{Q}})}$$

Moore, 2007:

$$\text{black holes} \longleftrightarrow \text{elliptic curves}/\overline{\mathbf{Q}}$$

Today

$$\boxed{(\text{phys.quant. in a point } P) = \int_{P_0}^{P} \omega}$$

for appropriate $\omega \in \Omega^1(\mathbf{V})$

1. Two-body problem: the classical study over $\mathbb{R}$

Kepler-Newton:

$$\begin{cases} \ddot{x} = -\gamma \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}} \\ \ddot{y} = -\gamma \dfrac{y}{(x^2+y^2)^{\frac{3}{2}}}. \end{cases}$$

Consider this system of ODE's in the phase space

$$\mathcal{P} := \mathrm{Spec}(\mathbb{R}[x, y, \dot{x}, \dot{y}]) \setminus \{(0,0,0,0)\}.$$

The *sector velocity integral*

$$\Sigma := x\dot{y} - \dot{x}y$$

fibers $\mathcal{P}$ in the integral quadrics (Σ-levels)

$$\mathcal{P} = \coprod_{\sigma \in \mathbb{R}} \mathbf{Q}_{\Sigma}.$$

Non-catastrophic solutions $(\dot{x}, \dot{y}) \notin \mathbb{R} \cdot (x, y)$:

$$\mathbb{R} \longrightarrow \mathcal{P} \setminus \mathbf{Q}_0.$$

$1'$. Two-body problem: the classical study over $\mathbb{R}$

In the polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ the Kepler-Newton system implies

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\gamma}{r^2},$$

while the sector velocity integral results in $\dot{\varphi} = \frac{\Sigma}{r^2}$, so the variables separate:

$$\boxed{\ddot{r} = \frac{\Sigma^2}{r^3} - \frac{\gamma}{r^2}}$$

Solutions of this ODE $\leftrightarrow$ integral curves of *rational* vector fields

$$\dot{r}\frac{\partial}{\partial r} + (\frac{\Sigma}{r^3} - \frac{\gamma}{r^2})\frac{\partial}{\partial \dot{r}}$$

on the $(r, \dot{r})$-affine plane; can be considered over *any* field.

$1''$. Two-body problem: the classical study over $\mathbb{R}$

The above equation $\ddot{r} = \frac{\Sigma^2}{r^3} - \frac{\gamma}{r^2}$ admits a further rational integral ("energy"...)

$$\boxed{E := \frac{\dot{r}^2}{2} + \frac{\Sigma^2}{2r^2} - \frac{\gamma}{r}}.$$

Using $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}$, rewrite it as

$$\mathrm{d}t = \frac{r\mathrm{d}r}{\sqrt{2Er^2 + 2\gamma r - \Sigma^2}}.$$

Is the 2-body problem solved? Well,

$$t - t_0 = \int \frac{r\mathrm{d}r}{\sqrt{2Er^2 + 2\gamma r - \Sigma^2}}.$$

We wanted $r(t)$, but got $t(r)$ as a nasty multi-valued expression (the integral will be calculated below). And this answer makes sense only $/\mathbb{R}$.

Conclusion. Trying to parametrize everything by t fails!

2. The angular "time"

The angle φ behaves better! According to the above and using $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\Sigma}{r^2}$, we arrive at

$$\boxed{\mathrm{d}\varphi = \frac{\Sigma \cdot \mathrm{d}r}{r\sqrt{2Er^2 + 2\gamma r - \Sigma^2}}},$$

that integrates to

$$\varphi - \varphi_0 = \arccos \frac{\frac{1}{r} - \frac{\gamma}{\Sigma^2}}{\sqrt{\frac{2E}{\Sigma^2} + \frac{\gamma^2}{\Sigma^4}}} \Longrightarrow r = \frac{\frac{\Sigma^2}{\gamma}}{1 + \sqrt{\frac{2\Sigma^2 E}{\gamma^2} + 1}\cos(\varphi - \varphi_0)}$$

Introducing $r_0 := \frac{\Sigma^2}{\gamma}$ and $\varepsilon := \sqrt{1 + \frac{2E\Sigma^2}{\gamma^2}}$, we get an *ellipse*

$$\boxed{r = \frac{r_0}{1 + \varepsilon\cos(\phi - \phi_0)}}.$$

Earth:

$$r_0 \approx 150\ 000\ 000\ \text{km},\ \ \varepsilon \approx .017$$

$2'$. The angular "time"

In terms of the initial phase space $\mathcal{P}$

$$x = \frac{r_0 \cos\varphi}{1+\varepsilon\cos(\varphi-\varphi_0)},$$

$$y = \frac{r_0 \sin\varphi}{1+\varepsilon\cos(\varphi-\varphi_0)},$$

$$\dot{x} = \sqrt{\frac{\gamma}{r_0}}(-\sin\varphi - \varepsilon\sin\varphi_0),$$

$$\dot{y} = \sqrt{\frac{\gamma}{r_0}}(\cos\varphi + \varepsilon\cos\varphi_0).$$

Is it the ultimate answer? YES – if φ is a TIME.

But **what** is a TIME? Mathematician's answer:

$$\boxed{\mathbb{T} : \mathcal{P}}.$$

2″. The angular "time"

Consider only the *nondimensionalized* coordinates of the (φ-dependent) position of the *second body*:

$$\frac{x}{r_0} = \frac{\cos\varphi}{1+\varepsilon\cos(\varphi-\varphi_0)},$$

$$\frac{y}{r_0} = \frac{\sin\varphi}{1+\varepsilon\cos(\varphi-\varphi_0)}.$$

Suppose $\varepsilon \in \mathbb{Q}$ and $0 \le \varepsilon < 1$. Then

$$\varphi, \varphi_0 \in \mathbb{R} \Longrightarrow \frac{x}{r_0}, \frac{y}{r_0} \in \mathbb{R};$$

$$\varphi, \varphi_0 \in \mathbb{Q}\pi \Longrightarrow \frac{x}{r_0}, \frac{y}{r_0} \in \mathbb{Q}^{\mathsf{ab}} \bigcap \mathbb{R};$$

$$N \in \mathbb{N},\ \varphi, \varphi_0 \in \mathbb{Z}\frac{2\pi}{N} \Longrightarrow \frac{x}{r_0}, \frac{y}{r_0} \in \mathbb{Q}(\sqrt[N]{1}) \bigcap \mathbb{R}.$$

The last two cases compatible with ALL the celestial observations!

$2'''$. The angular "time"

Trying to do the same with the *physical* time t:

$$t - t_0 = \int \frac{\mathrm{d}r}{\sqrt{\gamma}\sqrt{\frac{\varepsilon^2}{r_0} - (\frac{\sqrt{r_0}}{r} - \frac{1}{\sqrt{r_0}})^2}}.$$

This integral can be calculated in elementary functions:

$$t - t_0 = \frac{(k+1)^2}{4k}[\frac{(k-1)z}{kz^2+1} + \sqrt{k}(k+1)\arctan(\sqrt{k}z)],$$

where $k = \frac{1+\varepsilon}{1-\varepsilon}$ and $z = \tan\frac{\varphi}{2}$.

But **no chances** to express r or φ in terms of t!

3. Two-body problem over an arbitrary field

The following pairs can provide "solutions" of the 2-body problem:

Group $\mathbb{T}$	**Field**
$\mathbb{R}$	$\mathbb{R}$
$\frac{\mathbb{R}}{2\pi\mathbb{Z}}$	$\mathbb{R}$
$\frac{\mathbb{Z}}{N\mathbb{Z}}$	$\mathbb{k} = \overline{\mathbb{k}}, \mathrm{char}(\mathbb{k}) \nmid N$
$\mathbb{Z}_p$	$\mathbb{k} = \overline{\mathbb{k}}, \mathrm{char}(\mathbb{k}) \neq p$
$\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$	$\mathbb{k} = \overline{\mathbb{k}}, \mathrm{char}(\mathbb{k}) = 0$

Lots of possibilities!

A **problem** over $\overline{\mathbb{Q}}$ – the idea of Yu. I. Manin: characterize "solutions" by the behavior of heights along the orbits (minimal oscillations? ..?)

4. Elliptic ”time”

The main parameter of the orbit:

$$k := \frac{1+\varepsilon}{1-\varepsilon}.$$

The main variable:

$$z := \tan\frac{\varphi}{2}.$$

The *divine* curve (imaginary *Legendre*)

$$\boxed{w^2 = (1+z^2)(1+k^2z^2)}$$

- depends on the planet and is defined by its orbit;
- is dimensionless;
- is nowhere (not embedded in the physical space);
- governs ”all” the observable variables related to the planet.

$4'$. Elliptic "time"

$$\boxed{w^2 = (1+z^2)(1+k^2z^2)}$$

Differentials:

$$\mathrm{d}\varphi = \frac{2\mathrm{d}z}{1+z^2};$$

$$\frac{\Sigma}{r_0^2}\mathrm{d}t = \frac{(k+1)^2}{2}\frac{1+z^2}{(1+kz^2)^2}\mathrm{d}z;$$

$$\frac{\mathrm{d}r}{r_0} = -(k^2-1)\frac{z}{(1-kz^2)^2}\mathrm{d}z;$$

$(\mathrm{d}s)^2 := (\dot{x}^2 + \dot{x}^2)(\mathrm{d}t)^2$

$$\frac{\mathrm{d}s}{r_0} = (k+1)\frac{w\mathrm{d}z}{(1+kz^2)^2};$$

$$\cdots.$$

Fact. *All the poles of all the (differentials of) quantities are in the points of 4th order.*

5. p-adic and adelic fantasies

Can this toy example be extended?

Turning to ≥ 3-body problems, we may try to introduce the *interaction* of divine curves (whatever it means).

In an analogy with Sh2004 (Proc. of this conference) one can hope that the chaotic behavior of the $\overline{\mathbb{Q}}$-models will occur in p-directions only for finite number of p's, so some *adelic* measure of chaos will appear.

The above mentioned 4-order points (2-isogenies, Landin transforms,...) suggests special consideration of **2-adic time.** A simple 2-adic model of period-doubling onset of chaos was considered in Dremov+Sh+ Vytnova2006 (Proc. of this conference).

6. Hitchin's ASD Einstein metrics, etc.

Another **example:** turn to the adult math.

Let $\Omega^1(\mathbf{S}^3) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ satisfy $\begin{cases} \mathrm{d}\sigma_1 = \sigma_2 \wedge \sigma_3 \\ \mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1 \\ \mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2 \end{cases}$

Tod (1994) and Hitchin (1995) have found on $(0,1) \times \mathbf{S}^3$ a family of SU_2-invariant metrics

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}t)^{\otimes 2}}{t(1-t)} + \frac{\sigma_1^{\otimes 2}}{\Omega_1^2} + \frac{(1-t)\sigma_2^{\otimes 2}}{\Omega_2^2} + \frac{t\sigma_3^{\otimes 2}}{\Omega_3^2},$$

where $\begin{cases} \Omega_1^2 = \frac{(x-t)^2 x(x-1)}{t(1-t)}[z - \frac{1}{2(x-1)}](z - \frac{1}{2x}) \\ \Omega_2^2 = \frac{x^2(x-1)(x-t)}{t}[z - \frac{1}{2(x-t)}][z - \frac{1}{2(x-1)}] \\ \Omega_3^2 = \frac{(x-1)^2 x(x-t)}{1-t}[(z - \frac{1}{2x}][z - \frac{1}{2(x-t)}] \end{cases}$,

the functions $x(t)$ and $z(t)$ defined in the next slide.

6′. Hitchin's ASD Einstein metrics, etc.

$x(t)$ solves Painlevé-VI (with parameters $\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}$)

$$\ddot{x} = \frac{1}{2}(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t})\dot{x}^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t})\dot{x} +$$

$$+ \frac{x(x-1)(x-t)}{t^2(t-1)^2}[\frac{1}{8} - \frac{t}{8x^2} + \frac{t-1}{8(x-1)^2} + \frac{3x(x-1)}{8(x-t)^2}]$$

and $z(t)$ is defined from

$$\dot{x} = \frac{x(x-1)(x-t}{t(t-1)}[2z - \frac{1}{2x} - \frac{1}{2(x-1)} + \frac{1}{2(x-t)}].$$

Everything expressible in terms of theta-functions!

$6''$. Hitchin's ASD Einstein metrics, etc.

Einstein $\leftrightarrow$ Painlevé, Tod's way.
The metric $(\mathrm{d}s)^2 = f(t)(\mathrm{d}t)^{\otimes 2} + a_1(t)\sigma_1^{\otimes 2} + a_2(t)\sigma_2^{\otimes 2} + a_3(t)\sigma_3^{\otimes 2}$ is ASD iff can be rescaled to

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}t)^{\otimes 2}}{t(1-t)} + \frac{\sigma_1^{\otimes 2}}{\Omega_1^2} + \frac{(1-t)\sigma_2^{\otimes 2}}{\Omega_2^2} + \frac{t\sigma_3^{\otimes 2}}{\Omega_3^2},$$

where the scalar functions $\Omega_{1,2,3}$ satisfy $\begin{cases} \dot{\Omega}_1 = -\frac{\Omega_2\Omega_3}{t(1-t)} \\ \dot{\Omega}_2 = -\frac{\Omega_3\Omega_1}{t} \\ \dot{\Omega}_3 = -\frac{\Omega_1\Omega_2}{1-t} \end{cases}$, known (Tod 1992) to reduce to $\mathrm{P}6_{\frac{1}{8},\frac{1}{8},c,d}$ with $c+d=\frac{1}{2}$; Einstein $\Longrightarrow c=\frac{1}{8}$.

Einstein $\leftrightarrow$ Painlevé, Hitchin's way. Twistors, families of rational curves on complex 3-manifolds, flat connections with log-singularities, isomonodromic deformations, Schlessinger equations, …

$6'''$. Hitchin's ASD Einstein metrics, etc.

Where is arithmetic? The most interesting solutions of Painleve-VI are *algebraic* – providing Einstein metrics defined by finite amounts of information.

Any such solution $x(t)$ has an *affine model* – a plane curve, defined by $F(t, x) = 0$. The "time" t is a *Belyi function* on this curve; hence, everything is defined over $\overline{\mathbb{Q}}$.

Hitchin (2004) constructs algebraic solutions using the Poncelet closure theorem; the dynamic governed by the finite-order points of elliptic curves, appears again!

What about p-adics? See Yi. I. Manin's talk.

Thank you!