On the elliptic time in the Newton-Kepler adelic dynamic

George SHABAT

(IUM, MSU, RSUH, ITEP; Moscow)
Plan of the talk

0. Physical models: transcendental vs. algebraic; \( \mathbb{Q} \) emerging
1. Two-body problem: the classical study over \( \mathbb{R} \)
2. The angular ”time”
3. Two-body problem over an arbitrary field
4. Elliptic ”time”
5. \( p \)-adic and adelic fantasies
0. Physical models: transcendental vs. algebraic

Algebro-geometric nature – often $\sqrt{Q}$ – of seemingly transcendent objects

Random examples (not for today’s talk)

\begin{align*}
\text{Voevodsky-Shabat, 1987:} \\
\int \cdots \int_{\text{Met}(S_g)} \leftrightarrow \sum_{\Delta_g} \leftrightarrow \sum_{\text{BelPairs}_g(Q)}
\end{align*}

Moore, 2007:

black holes $\leftrightarrow$ elliptic curves $/Q$

Today

\begin{align*}
(\text{phys.quant. in a point } P) &= \int_{P_0}^{P} \omega \\
\text{for appropriate } \omega \in \Omega^1(V)
\end{align*}
1. Two-body problem: the classical study over $\mathbb{R}$

Kepler-Newton:

$$\begin{cases}
\ddot{x} = -\gamma \frac{x}{(x^2 + y^2)^{3/2}} \\
\ddot{y} = -\gamma \frac{y}{(x^2 + y^2)^{3/2}}.
\end{cases}$$

Consider this system of ODE's in the phase space $\mathcal{P} := \text{Spec}(\mathbb{R}[x, y, \dot{x}, \dot{y}]) \setminus \{(0, 0, 0, 0)\}$.

The sector velocity integral

$$\Sigma := xy - \dot{x}y$$

fibers $\mathcal{P}$ in the integral quadrics ($\Sigma$-levels)

$$\mathcal{P} = \bigsqcup_{\sigma \in \mathbb{R}} \mathcal{Q}_\Sigma.$$

Non-catastrophic solutions $(\dot{x}, \dot{y}) \notin \mathbb{R} \cdot (x, y)$:

$$\mathbb{R} \rightarrow \mathcal{P} \setminus \mathcal{Q}_0.$$
1. Two-body problem: the classical study over $\mathbb{R}$

In the polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$ the Kepler-Newton system implies

$$\ddot{r} - r \dot{\varphi}^2 = -\frac{\gamma}{r^2},$$

while the sector velocity integral results in $\dot{\varphi} = \frac{\Sigma}{r^2}$, so the variables separate:

$$\ddot{r} = \frac{\Sigma^2}{r^3} - \frac{\gamma}{r^2}$$

Solutions of this ODE $\leftrightarrow$ integral curves of rational vector fields

$$\dot{r} \frac{\partial}{\partial r} + \left(\frac{\Sigma}{r^3} - \frac{\gamma}{r^2}\right) \frac{\partial}{\partial \dot{r}}$$

on the $(r, \dot{r})$-affine plane; can be considered over any field.
Two-body problem: the classical study over $\mathbb{R}$

The above equation $\ddot{r} = \frac{\Sigma^2}{r^3} - \frac{\gamma}{r^2}$ admits a further rational integral ("energy"...)

$$E := \frac{\dot{r}^2}{2} + \frac{\Sigma^2}{2r^2} - \frac{\gamma}{r}.$$

Using $\dot{r} = \frac{dr}{dt}$, rewrite it as

$$dt = \frac{rdr}{\sqrt{2Er^2 + 2\gamma r - \Sigma^2}}.$$

Is the 2-body problem solved? Well,

$$t - t_0 = \int \frac{rdr}{\sqrt{2Er^2 + 2\gamma r - \Sigma^2}}.$$

We wanted $r(t)$, but got $t(r)$ as a nasty multi-valued expression (the integral will be calculated below). And this answer makes sense only $\in \mathbb{R}$.

Conclusion. Trying to parametrize everything by $t$ fails!
2. The angular ”time”

The angle $\varphi$ behaves better! According to the above and using $\frac{d\varphi}{dt} = \frac{\sum}{r^2}$, we arrive at

$$d\varphi = \frac{\sum \cdot dr}{r \sqrt{2Er^2 + 2\gamma r - \sum^2}},$$

that integrates to

$$\varphi - \varphi_0 = \arccos \frac{1}{r} - \frac{\gamma}{\sum^2} \implies r = \frac{\sum^2}{\gamma} \sqrt{\frac{2E}{\sum^2} + \frac{\gamma^2}{\sum^4}} \cdot 1 + \sqrt{\frac{2\sum^2 E}{\gamma^2}} + 1 \cos(\varphi - \varphi_0)$$

Introducing $r_0 := \frac{\sum^2}{\gamma}$ and $\varepsilon := \sqrt{1 + \frac{2E\sum^2}{\gamma^2}}$, we get an ellipse

$$r = \frac{r_0}{1 + \varepsilon \cos(\phi - \phi_0)}.$$

Earth:

$$r_0 \approx 150 000 000 \text{ km}, \quad \varepsilon \approx .017$$
2'. The angular "time"

In terms of the initial phase space \( \mathcal{P} \)

\[
x = \frac{r_0 \cos \varphi}{1 + \varepsilon \cos(\varphi - \varphi_0)},
\]

\[
y = \frac{r_0 \sin \varphi}{1 + \varepsilon \cos(\varphi - \varphi_0)},
\]

\[
\dot{x} = \sqrt{\frac{\gamma}{r_0}} (- \sin \varphi - \varepsilon \sin \varphi_0),
\]

\[
\dot{y} = \sqrt{\frac{\gamma}{r_0}} (\cos \varphi + \varepsilon \cos \varphi_0).
\]

Is it the ultimate answer? YES – if \( \varphi \) is a TIME.

But what is a TIME? Mathematician’s answer:

\[ \mathbb{T} : \mathcal{P}. \]
2". The angular "time"

Consider only the nondimensionalized coordinates of the (φ-dependent) position of the second body:

\[
\frac{x}{r_0} = \cos \varphi \frac{\cos \varphi}{1 + \varepsilon \cos(\varphi - \varphi_0)},
\]

\[
\frac{y}{r_0} = \sin \varphi \frac{\sin \varphi}{1 + \varepsilon \cos(\varphi - \varphi_0)}.
\]

Suppose \( \varepsilon \in \mathbb{Q} \) and \( 0 \leq \varepsilon < 1 \). Then

\[
\varphi, \varphi_0 \in \mathbb{R} \implies \frac{x}{r_0}, \frac{y}{r_0} \in \mathbb{R};
\]

\[
\varphi, \varphi_0 \in \mathbb{Q} \pi \implies \frac{x}{r_0}, \frac{y}{r_0} \in \mathbb{Q}^{ab} \bigcap \mathbb{R};
\]

\[
N \in \mathbb{N}, \ \varphi, \varphi_0 \in \mathbb{Z} \frac{2\pi}{N} \implies \frac{x}{r_0}, \frac{y}{r_0} \in \mathbb{Q}(\sqrt{\frac{N}{1}}) \bigcap \mathbb{R}.
\]

The last two cases compatible with ALL the celestial observations!
2″. The angular ”time”

Trying to do the same with the physical time $t$:

$$t - t_0 = \int \frac{dr}{\sqrt{\gamma} \sqrt{\frac{\varepsilon^2}{r_0} - \left(\frac{\sqrt{r_0}}{r} - \frac{1}{\sqrt{r_0}}\right)^2}}.$$

This integral can be calculated in elementary functions:

$$t - t_0 = \frac{(k + 1)^2}{4k} \left[ \frac{(k - 1)z}{kz^2 + 1} + \sqrt{k(k + 1)} \arctan(\sqrt{k}z) \right],$$

where $k = \frac{1 + \varepsilon}{1 - \varepsilon}$ and $z = \tan \frac{\varphi}{2}$.

But no chances to express $r$ or $\varphi$ in terms of $t$!
### 3. Two-body problem over an arbitrary field

The following pairs can provide ”solutions” of the 2-body problem:

<table>
<thead>
<tr>
<th>Group $\mathbb{T}$</th>
<th>Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2\pi\mathbb{Z}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>$\mathbb{Z}/N\mathbb{Z}$</td>
<td>$\mathbb{k} = \overline{k}$, $\text{char}(k) \nmid N$</td>
</tr>
<tr>
<td>$\mathbb{Z}_p$</td>
<td>$\mathbb{k} = \overline{k}$, $\text{char}(k) \neq p$</td>
</tr>
<tr>
<td>$\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$</td>
<td>$\mathbb{k} = \overline{k}$, $\text{char}(k) = 0$</td>
</tr>
</tbody>
</table>

Lots of possibilities!

A *problem* over $\overline{\mathbb{Q}}$ – the idea of Yu. I. Manin: characterize ”solutions” by the behavior of heights along the orbits (minimal oscillations? ..?)
4. Elliptic “time”

The main parameter of the orbit:

\[ k := \frac{1 + \varepsilon}{1 - \varepsilon}. \]

The main variable:

\[ z := \tan \frac{\varphi}{2}. \]

The *divine* curve (imaginary *Legendre*)

\[ w^2 = (1 + z^2)(1 + k^2 z^2) \]

• depends on the planet and is defined by its orbit;
• is dimensionless;
• is nowhere (not embedded in the physical space);
• governs ”all” the observable variables related to the planet.
4'. Elliptic "time"

\[ w^2 = (1 + z^2)(1 + k^2 z^2) \]

Differentials:

\[ d\varphi = \frac{2dz}{1 + z^2}; \]
\[ \sum \frac{dt}{r_0^2} = \frac{(k + 1)^2}{2} \frac{1 + z^2}{(1 + k^2 z^2)^2} dz; \]
\[ \frac{dr}{r_0} = -(k^2 - 1) \frac{z}{(1 - k^2 z^2)^2} dz; \]
\[ (ds)^2 := (\dot{x}^2 + \dot{z}^2)(dt)^2 \]
\[ \frac{ds}{r_0} = (k + 1) \frac{wdz}{(1 + k^2 z^2)^2}; \]

Fact. All the poles of all the (differentials of) quantities are in the points of 4th order.
5. $p$-adic and adelic fantasies

Can this toy example be extended?

Turning to $\geq 3$-body problems, we may try to introduce the interaction of divine curves (whatever it means).

In an analogy with Sh2004 (Proc. of this conference) one can hope that the chaotic behavior of the $\overline{\mathbb{Q}}$-models will occur in $p$-directions only for finite number of $p$’s, so some adelic measure of chaos will appear.

The above mentioned 4-order points (2-isogenies, Landin transforms,...) suggests special consideration of **2-adic time**. A simple 2-adic model of period-doubling onset of chaos was considered in Dremov+Sh+Vytnova2006 (Proc. of this conference).
6. Hitchin’s ASD Einstein metrics, etc.

Another example: turn to the adult math.

Let $\Omega^1(S^3) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ satisfy

\[
\begin{align*}
    d\sigma_1 &= \sigma_2 \wedge \sigma_3 \\
    d\sigma_2 &= \sigma_3 \wedge \sigma_1 \\
    d\sigma_3 &= \sigma_1 \wedge \sigma_2
\end{align*}
\]

Tod (1994) and Hitchin (1995) have found on $(0, 1) \times S^3$ a family of $SU_2$-invariant metrics

\[
(ds)^2 = \frac{(dt)^{\otimes 2}}{t(1-t)} + \frac{\sigma_1^{\otimes 2}}{\Omega_1^2} + \frac{(1-t)\sigma_2^{\otimes 2}}{\Omega_2^2} + \frac{t\sigma_3^{\otimes 2}}{\Omega_3^2},
\]

where

\[
\begin{align*}
    \Omega_1^2 &= \frac{(x-t)^2x(x-1)}{t(1-t)} [z - \frac{1}{2(x-1)}] (z - \frac{1}{2x}) \\
    \Omega_2^2 &= \frac{x^2(x-1)(x-t)}{t} [z - \frac{1}{2(x-t)}] [z - \frac{1}{2(x-1)}] \\
    \Omega_3^2 &= \frac{(x-1)^2x(x-t)}{1-t} [(z - \frac{1}{2x}) [z - \frac{1}{2(x-t)}]
\end{align*}
\]

the functions $x(t)$ and $z(t)$ defined in the next slide.
6'. Hitchin’s ASD Einstein metrics, etc.

$x(t)$ solves Painlevé-VI (with parameters $\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}$)

$$\ddot{x} = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x - t} \right) \dot{x}^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{x - t} \right) \dot{x} +$$

$$+ \frac{x(x - 1)(x - t)}{t^2(t - 1)^2} \left[ \frac{1}{8} - \frac{t}{8x^2} + \frac{t - 1}{8(x - 1)^2} + \frac{3x(x - 1)}{8(x - t)^2} \right]$$

and $z(t)$ is defined from

$$\dot{x} = \frac{x(x - 1)(x - t)}{t(t - 1)} \left[ 2z - \frac{1}{2x} - \frac{1}{2(x - 1)} + \frac{1}{2(x - t)} \right].$$

Everything expressible in terms of theta-functions!
Einstein ↔ Painlevé, Tod’s way.
The metric \((ds)^2 = f(t)(dt)\otimes^2 + a_1(t)\sigma_1^\otimes^2 + a_2(t)\sigma_2^\otimes^2 + a_3(t)\sigma_3^\otimes^2\)
is ASD iff can be rescaled to

\[
(ds)^2 = \frac{(dt)^\otimes^2}{t(1-t)} + \frac{\sigma_1^\otimes^2}{\Omega_1^2} + \frac{(1-t)\sigma_2^\otimes^2}{\Omega_2^2} + \frac{t\sigma_3^\otimes^2}{\Omega_3^2},
\]

where the scalar functions \(\Omega_{1,2,3}\) satisfy

\[
\begin{align*}
\dot{\Omega}_1 &= -\frac{\Omega_2\Omega_3}{t(1-t)} \\
\dot{\Omega}_2 &= -\frac{\Omega_3\Omega_1}{t} \\
\dot{\Omega}_3 &= -\frac{\Omega_1\Omega_2}{1-t}
\end{align*}
\]

(Tod 1992) to reduce to \(\text{P}6_{\frac{1}{8}, \frac{1}{8}, c, d}\) with \(c + d = \frac{1}{2}\); Einstein \(\implies c = \frac{1}{8}\).

Einstein ↔ Painlevé, Hitchin’s way. Twistors, families of rational curves on complex 3-manifolds, flat connections with log-singularities, isomonodromic deformations, Schlessinger equations, ...
Where is arithmetic? The most interesting solutions of Painleve-VI are algebraic – providing Einstein metrics defined by finite amounts of information.

Any such solution $x(t)$ has an affine model – a plane curve, defined by $F(t, x) = 0$. The ”time” $t$ is a Belyi function on this curve; hence, everything is defined over $\overline{\mathbb{Q}}$.

Hitchin (2004) constructs algebraic solutions using the Poncelet closure theorem; the dynamic governed by the finite-order points of elliptic curves, appears again!

What about p-adics? See Yi. I. Manin’s talk.

Thank you!