T-duality in string theory

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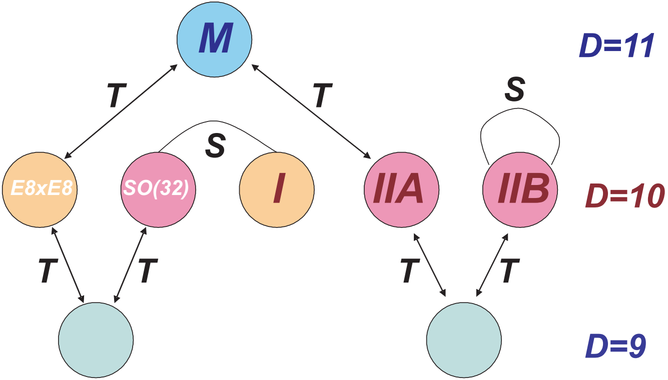
Motivation

- String theory should
  provide consistent quantum gravity
  Unify all interaction in a single theory
- In particle physics there are infinity manu different theories
- Consistency conditions (absence of anomalies) leads to finite number of superstring theories

Consistent string theories and M theory



Connected by web of dualities

- There is not yet satisfactory formulation of M-theory

Action of the bosonic string

- Action for the closed string in the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$

$$S[x] = \kappa \int_{\Sigma} d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma,$$

  where

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}.$$

  - $G_{\mu\nu} = G_{\nu\mu}$ metric tensor
  - $B_{\mu\nu} = -B_{\nu\mu}$ Kalb-Ramond field

Choice of the background

- consistency of the theory implies conformal invariance on the quantum level
- space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0$$

$$D_{\rho} B^{\rho}{}_{\mu\nu} = 0$$

- particular solution

$$G_{\mu\nu}[x] = const \qquad B_{\mu\nu}[x] = const$$

Standard Bouscher's construction of T-duality
Gauging the symmetry

- If background fields are x^a-independent, the action is invariant under the global shift symmetry $\delta x^\mu = \varepsilon^\mu = const$
- To localize this symmetry $\delta x^\mu = \varepsilon^\mu(\tau, \sigma)$, we introduce the gauge fields v_α^μ and substitute the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu$$

Standard Bouscher's construction of T-duality
Gauge invariant action

- Dual theory should be equivalent to the initial one.
- To destroy new degrees of freedom originating from gauge fields we require that field strength vanish

$$F^{\mu}_{\alpha\beta} \equiv \partial_{\alpha} v^{\mu}_{\beta} - \partial_{\beta} v^{\mu}_{\alpha}$$

- This is achieved by introducing the Lagrange multiplier y_{μ}, and the appropriate term in the Lagrangian

$$S_{inv} = \kappa \int d^2\xi \Big[D_{+}x^{\mu} \Pi_{+\mu\nu} D_{-}x^{\nu} + \frac{1}{2} y_{\mu} F^{\mu}_{+-} \Big],$$

Standard Bouscher's construction of T-duality
Gauge fixed action

- We fix the gauge $x^\mu = 0$ and obtain gauge fixed action

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}(v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu) \Big]$$

where y_μ and $v_\pm^\mu$ are independent variables

- Equations of motion with respect to y_μ

$$\partial_\alpha v_\beta^\mu - \partial_\beta v_\alpha^\mu = 0 \quad \Longrightarrow \quad v_\pm^\mu = \partial_\pm x^\mu$$

produce the initial action

Standard Bouscher's construction of T-duality
T-dual action

- The T-dual action will be obtained by integrating out the gauge fields from the gauge fixed action.
- The equations of motion with respect to the gauge fields $v_\pm^\mu$ are

$$\Pi_{\mp\mu\nu} v_\pm^\nu + \frac{1}{2}\partial_\pm y_\mu = 0 \quad \Longrightarrow \quad v_\pm^\mu = -\kappa\, \Theta_\pm^{\mu\nu} \partial_\pm y_\nu$$

Here

$$\Theta_\pm^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu},$$

and $G^E_{\mu\nu} \equiv [G - 4BG^{-1}B]_{\mu\nu}$, $\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ are the effective metric and the non-commutativity parameter

Standard Bouscher's construction of T-duality
T-dual action 2

- Substituting equation of motion into the gauge fixed action, we obtain T-dual action

$$^\star S[y] \equiv S_{fix}[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu$$

- Initial action transforms into the T-dual action under

$$\partial_\pm x^\mu \to \partial_\pm y_\mu$$

$$\begin{aligned} \Pi_{+\mu\nu} &\to {}^\star\Pi_{+\mu\nu} = \frac{\kappa}{2}\Theta_-^{\mu\nu} \\ G_{\mu\nu} &\to {}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu} \\ B_{\mu\nu} &\to {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu} \end{aligned}$$

Example 1- One compact dimension

- x^{25} is compact

$$\mu = \nu = 25\,, \qquad B_{\mu\nu} = B_{25,25} = 0\,,$$

Dual fields

$$^{\star}G^{\mu\nu} = (\alpha')^2 (G^{-1})^{\mu\nu}\,, \quad ^{\star}B^{\mu\nu} = 0$$

- Duality in terms of radius

$$2\pi R = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{G ds^2} = 2\pi\sqrt{G}$$

$$G = R^2 \ , \quad ^{\star}G = {^{\star}R^2} \quad \to \quad ^{\star}R = \frac{\alpha'}{R}$$

Example 2- Arbitrary number of compact dimensions

- Chain of T-duality

$$S(x) \xrightarrow{\mathrm{T}^1} S_1 \xrightarrow{\mathrm{T}^2} \cdots \xrightarrow{\mathrm{T}^{\mathrm{d}}} S_d \xrightarrow{\mathrm{T}^{\mathrm{d+1}}} S_{d+1} \xrightarrow{\mathrm{T}^{(\mathrm{d+2})}} \cdots \xrightarrow{\mathrm{T}^{\mathrm{D}}} S_D(y)$$

- After same procedure, but with a complicated calculation

$$ {}_a\Pi_{\pm}^{ab} = \frac{\kappa}{2}\hat{\theta}_{\mp}^{ab}, \qquad {}_a\Pi^{a}_{\pm\,i} = \kappa\hat{\theta}_{\mp}^{ab}\Pi_{\pm bi}$$

$$ {}_a\Pi_{\pm i}{}^{a} = -\kappa\Pi_{\pm ib}\hat{\theta}_{\mp}^{ba}, \qquad {}_a\Pi_{\pm ij} = \Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\hat{\theta}_{\mp}^{ab}\Pi_{\pm bj}$$

where

$$a = 0, 1, \cdots, d-1, \qquad i = d, d+1, \cdots, D-1$$

Standard Bouscher's construction of T-duality
T-dual transformation rules

- T-dual transformations

$$v_{\pm}^{\mu} \cong \partial_{\pm} x^{\mu} \cong -\kappa \Theta_{\pm}^{\mu\nu} \partial_{\pm} y_{\nu}$$

- together with the inverse transformation

$$\partial_{\pm} x^{\mu} \cong -\kappa \Theta_{\pm}^{\mu\nu} \partial_{\pm} y_{\nu} \,,$$

$$\partial_{\pm} y_{\mu} \cong -2\Pi_{\mp\mu\nu} \partial_{\pm} x^{\nu} \,,$$

- T-dual transformation and its inverse

$$\pm\partial_{\pm} y = G_E \partial_{\pm} x - 2(BG^{-1})\partial_{\pm} y$$

$$\pm\partial_{\pm} x = 2(G^{-1}B)\partial_{\pm} x + G^{-1}\partial_{\pm} y$$

T-duality in doubled space
Generalized metric

- T-dual transformation in doubled space

$$\partial_\pm Z^M \cong \pm\,\Omega^{MN}\mathcal{H}_{NK}\,\partial_\pm Z^K$$

with new coordinates Z^M

$$Z^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix}$$

where $\mathcal{H}_{MN}$ is generalized metric

$$\mathcal{H}_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}$$

and

$$\Omega^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

T-duality in doubled space
Dual generalized metric

- T-duality $\Longleftrightarrow$ replacement x^μ with y_μ

$$^\star Z^M = \begin{pmatrix} y_\mu \\ x^\mu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} = \mathcal{T} Z$$

- Require that T-dual transformation for double coordinates $^\star Z^M$ has the same form as initial one

$$\partial_\pm {}^\star Z^M \cong \pm\, \Omega^{MN\,\star}\mathcal{H}_{NK}\, \partial_\pm {}^\star Z^K$$

we find $\quad {}^\star\mathcal{H} = \mathcal{T}\mathcal{H}\mathcal{T}$

T-duality in doubled space
Along all coordinates

- T-dual generalized metric

$$
{}^{\star}\mathcal{H}_{MN} = \begin{pmatrix} (G^{-1})^{\mu\nu} & 2(G^{-1})^{\mu\rho}B_{\rho\nu} \\ -2B_{\mu\rho}(G^{-1})^{\rho\nu} & G^{E}_{\mu\nu} \end{pmatrix} =
$$

$$
\begin{pmatrix} {}^{\star}G^{E}_{\mu\nu} & -2{}^{\star}B_{\mu\rho}({}^{\star}G^{-1})^{\rho\nu} \\ 2({}^{\star}G^{-1})^{\mu\rho\star}B_{\rho\nu} & ({}^{\star}G^{-1})^{\mu\nu} \end{pmatrix}
$$

- T-dual background fields

$$
{}^{\star}G^{\mu\nu} = (G_E^{-1})^{\mu\nu}\,, \quad {}^{\star}B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}
$$

T-duality in doubled space
Along arbitrary subset of coordinates

- Along arbitrary number of coordinates $x^a, \; a = 0, 1, \cdots d-1$

$$Z_a^M = \mathcal{T}^{aM}{}_N Z^N \qquad \begin{pmatrix} y_a \\ x^i \\ x^a \\ y_i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1_a & 0 \\ 0 & 1_i & 0 & 0 \\ 1_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_i \end{pmatrix} \begin{pmatrix} x^a \\ x^i \\ y_a \\ y_i \end{pmatrix}$$

  Produce the same background fields as Buscher approach

- T-duality is the replacement of coordinates in doubled space, and must be non-physical

Type IIB superstring – Action

- Action for IIB superstring in pure spinor formulation

$$S(x^\mu, \theta^\alpha, \bar{\theta}^\alpha, \pi_\alpha, \bar{\pi}_\alpha) = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu$$

$$+ \int_\Sigma d^2\xi \left[-\pi_\alpha \partial_-(\theta^\alpha + \Psi^\alpha_\mu x^\mu) + \partial_+(\bar{\theta}^\alpha + \bar{\Psi}^\alpha_\mu x^\mu)\bar{\pi}_\alpha + \frac{1}{2\kappa} e^\phi \pi_\alpha F^{\alpha\beta} \bar{\pi}_\beta \right]$$

- Super coordinates $x^\mu, \theta^\alpha, \bar{\theta}^\alpha$
  $\pi_\alpha, \bar{\pi}_\alpha$ canonically conjugate momenta

Type IIB superstring – Background fields

- NS-NS sector: Gravitation $G_{\mu\nu}$
  antisymmetric Kalb-Ramond field $B_{\mu\nu}$
  Dilaton field Φ
- NS-R sector: two Gravitino fields Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$
  (Majorana-Weyl spinors of the same chirality)
- R-R sector: $F^{\alpha\beta}$ field strength
  can be expressed in terms of the antisymmetric tensors $F_{(k)}$

$$F^{\alpha\beta} = \sum_{k=0}^{D} \frac{1}{k!} F_{(k)} \Gamma_{(k)}^{\alpha\beta}, \quad \left[\Gamma_{(k)}^{\alpha\beta} = (C\Gamma^{[\mu_1 \dots \mu_k]})^{\alpha\beta} \right]$$

where

$$\Gamma^{[\mu_1 \mu_2 \dots \mu_k]} \equiv \Gamma^{[\mu_1} \Gamma^{\mu_2} \dots \Gamma^{\mu_k]}$$

Type IIB superstring
Compatibility between T-duality and supersymmetry

- Left and right world-sheet sectors transform differently under T-duality
- In T-dual theory there are two types of vielbeins, Γ-matrices, spin connections and supersymmetry transformations
- We should have a single geometry of T-dual theory
  All these different representations of the same variables can be connected by Lorentz transformations

Type IIB superstring
Compatibility between T-duality and supersymmetry 2

- Γ-matrices are related by the expression

$$ {}_a\bar{\Gamma}^{\hat{\mu}} = {}_a\Omega^{-1}\, {}_a\Gamma^{\hat{\mu}}\, {}_a\Omega $$

where ${}_a\Omega$ is spinorial representation of Lorentz transformation

$$ {}_a\Omega^{-1}\, \Gamma^{\underline{a}}\, {}_a\Omega = ({}_a\Lambda^{-1})^{\underline{a}}{}_{\underline{b}}\, \Gamma^{\underline{b}} $$

- With solution

$$ {}_a\Omega \; = \; {}_a\Gamma\, (i\, \Gamma_{11})^d \qquad {}_a\Gamma \equiv \Gamma^{\mu_1} \Gamma^{\mu_2} \dots \Gamma^{\mu_d} $$

- For $d = 1$

$$ {}_1\Omega \; = i\, \Gamma^{\mu}\, \Gamma_{11} $$

Type IIB superstring
Compatibility between T-duality and supersymmetry 3

- In order to have the same supersymmetry transformations in T-dual theory, we should chose new dual bar spinors

$$\overset{\bullet}{{}_a}\bar{\pi}_\alpha \equiv ({}_a\Omega^T)_\alpha{}^\beta \; {}_a\bar{\pi}_\beta \,, \qquad \overset{\bullet}{{}_a}\bar{\theta}^\alpha \equiv ({}_a\Omega)^\alpha{}_\beta \; \bar{\theta}^\beta$$

  They satisfies the same transformations as not overlaid ones

Relation between Type IIB and Type IIA superstring theories

- IIA: Gravitino fields Ψ^α_μ and $\bar\Psi^\alpha_\mu$ with **different** chiralities
  $\Gamma_{11} F^{\alpha\beta} = F^{\alpha\beta}\Gamma_{11} \implies F_{(k)}, \quad k = 2, 4$
- IIB: Gravitino fields Ψ^α_μ and $\bar\Psi^\alpha_\mu$ with **same** chirality
  $\Gamma_{11} F^{\alpha\beta} = -F^{\alpha\beta}\Gamma_{11} \implies F_{(k)}, \quad k = 1, 3, 5$
- $\overset{\bullet}{{}_a}\bar\Psi^{\alpha a} = \kappa\, {}_a\Omega^\alpha{}_\beta\, \hat\theta_-^{ab}\bar\Psi^\beta_b\,, \qquad {}_aF^{\alpha\beta} = {}_a\Omega^\alpha{}_\gamma F^{\gamma\beta}$

$$\Gamma_{11}\ {}_a\Omega = (-1)^d\ {}_a\Omega\ \Gamma_{11}$$

  for $d = 2n+1 \implies \Gamma_{11}\ {}_a\Omega = -\,{}_a\Omega\ \Gamma_{11}$
- T-duality along **odd** number of coordinates maps Type IIA and Type IIB theories to each other

Conclusion

- In double super space permutation of odd number of coordinates x^a with corresponding dual ones y_a maps Type IIA and Type IIB theories to each other
- Double super space frameworks unifies all theories T-dual to Type IIA and Type IIB
- We hope that inclusion of fermionic T-duality in double super space can lead to formulation of M theory