Fourier Transformation in the p-adic Langlands program

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p-ADICS 2015

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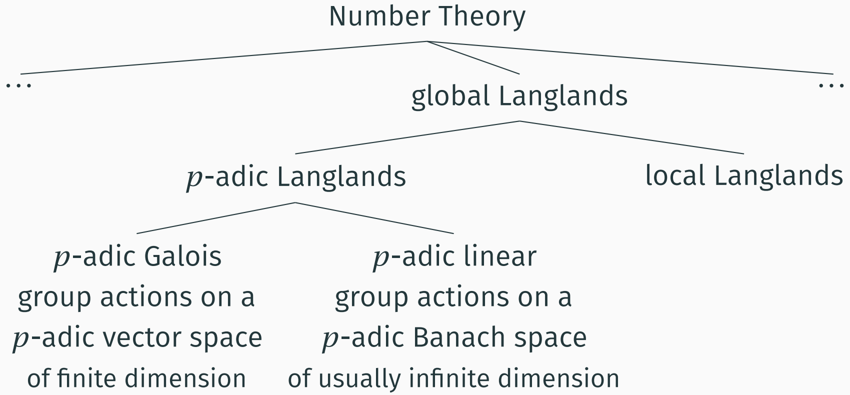
Number Theory
…
global Langlands
…
p-adic Langlands
local Langlands
p-adic Galois
group actions on a
p-adic vector space
of finite dimension
p-adic linear
group actions on a
p-adic Banach space
of usually infinite dimension

p-adic Langlands correspondence

p-adic vector space := vector space over (an extension of) $\mathbb{Q}_p$
p-adic Banach space := complete normed p-adic vector space

Definition

An action of a group G on a normed space with norm $\|\cdot\|$ is *unitary* if

$$\|g\cdot\| = \|\cdot\| \quad \text{for all } g \text{ in } \mathrm{G}.$$

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \text{ on} \\ p\text{-adic vector spaces} \\ \text{of dimension } n \end{array} \right\} \overset{?}{\leftrightarrow} \left\{ \begin{array}{l} \text{unitary continuous actions} \\ \text{of } \mathrm{GL}_n(\mathbb{Q}_p) \text{ on} \\ p\text{-adic Banach spaces} \\ \text{of (usually) infinite dimension} \end{array} \right\}$$

Functorial Construction

- $\mathscr{E}$:= a ring (in fact field) of p-adic power series in $X^{\pm 1}$
- *étale* φ, Γ*-module over* $\mathscr{E}$:= a module over $\mathscr{E}$ with a *semilinear* action of two commuting matrices φ and Γ

First

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \text{ on} \\ p\text{-adic vector spaces} \\ \text{of dimension } n \end{array} \right\} \overset{\sim}{\leftrightarrow} \left\{ \begin{array}{l} \text{étale } \varphi, \Gamma\text{-modules} \\ \text{over } \mathscr{E} \text{ of dimension } n \end{array} \right\}$$

then

$$\left\{ \begin{array}{l} \text{étale } \varphi, \Gamma\text{-modules} \\ \text{over } \mathscr{E} \text{ of dimension } n \end{array} \right\} \to \left\{ \begin{array}{l} \text{unitary continuous actions} \\ \text{of } \mathrm{GL}_n(\mathbb{Q}_p) \text{ on} \\ p\text{-adic Banach spaces} \\ \text{of (usually) infinite dimension} \end{array} \right\}$$

Cyclotomic Extension

Put

- $1, \zeta_p, \zeta_{p^2}, \ldots :=$ roots of unity of p-power order
- $\mathbb{Q}_p^{\mathrm{cyc}} := \mathbb{Q}_p(1, \zeta_p, \zeta_{p^2}, \ldots)$

Then

$$\overline{\mathbb{Q}}_p \underset{H\circlearrowleft}{- - -} \mathbb{Q}_p^{\mathrm{cyc}} \underset{\Gamma\circlearrowleft}{- - -} \mathbb{Q}_p$$

where

$$\Gamma := \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Z}_p^*$$

$$\sigma \mapsto x \text{ given by } \zeta^\sigma = \zeta^x \text{ for all } \zeta = 1, \zeta_p, \zeta_{p^2} \ldots$$

From characteristic 0 to p

Theorem (Field of Norms)

The absolute Galois groups of $\mathbf{F}_p((t))$ and $\mathbb{Q}_p^{\mathrm{cyc}}$ are isomorphic.

Put $\varphi :=$ Frobenius of $\mathbf{F}_p((t))$

Theorem

Let $\mathbf{E}$ be a field of characteristic p.

$$\left\{\begin{array}{l}\textit{continuous actions}\\ \textit{of } \mathrm{Gal}(\overline{\mathbf{E}}/\mathbf{E}) \textit{ on}\\ \textit{vector spaces over } \mathbb{F}_p\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\textit{semilinear injective actions}\\ \textit{of } \varphi \textit{ on vector spaces over } \mathbf{E}\end{array}\right\}$$

Corollary

Let $\mathscr{E}$:= *ring of* p*-adic power series in* $\mathrm{X}^{\pm 1}$ *lifting* $\mathbf{F}_p((t))$

$$\left\{\begin{array}{l}\textit{continuous actions}\\ \textit{of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{cyc}}) \textit{ on}\\ p\textit{-adic vector spaces}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\textit{semilinear injective actions}\\ \textit{of } \varphi \textit{ on vector spaces over } \mathscr{E}\end{array}\right\}$$

Proof.

By the preceding theorem using

- $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{cyc}}) \cong \mathrm{Gal}(\overline{\mathbf{F}_p((t))}/\mathbf{F}_p((t)))$, and
- lifting the vector space coefficients from $\mathbf{F}_p$ to $\mathbb{Q}_p$ by applying the functor of Witt vectors and inverting p. □

Theorem (Fontaine)

Let $\mathcal{E}$ *:= ring of* p*-adic power series in* $X^{\pm 1}$ *lifting* $\mathbf{F}_p((t))$ *with*

- $\varphi \circlearrowleft \mathcal{E}$ *by* $t^\varphi := (1+t)^p - 1$*, and*
- $\Gamma \circlearrowleft \mathcal{E}$ *by* $t^\gamma := (1+t)^\gamma - 1 = \sum \binom{\gamma}{n} t^n$ *where* $\Gamma \cong \mathbb{Z}_p^*$

$$\left\{ \begin{array}{l} \textit{continuous actions} \\ \textit{of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \textit{ on} \\ p\textit{-adic vector spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \textit{semilinear injective actions} \\ \textit{of commutative } \varphi \textit{ and } \Gamma \\ \textit{on vector spaces over } \mathcal{E} \end{array} \right\}$$

Proof.

By the preceding theorem for $H = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{cyc}})$ using

$$\overline{\mathbb{Q}}_p \underset{H\circlearrowleft}{- - -} \mathbb{Q}_p^{\mathrm{cyc}} \underset{\Gamma\circlearrowleft}{- - -} \mathbb{Q}_p.$$

Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathbf{o}_\mathbf{K}$. Denote

- $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K}) := \{$ all continuous $f\colon \mathbb{Z}_p \to \mathbf{K}\}$, and
- $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) := \{$ all continuous linear $\nu\colon \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}\}$.

Theorem

Then $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \mathbf{K} \otimes_{\mathbf{o}_\mathbf{K}} \mathbf{o}_\mathbf{K}[[X]]$ *as normed* $\mathbf{K}$*-algebras.*

Proof.

- By density of the locally constant functions, that is, $\mathbf{o}_\mathbf{K}[\mathbb{Z}/p\mathbb{Z}] \cup \mathbf{o}_\mathbf{K}[\mathbb{Z}/p^2\mathbb{Z}] \cup \ldots$ inside $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{o}_\mathbf{K})$

$$\mathscr{D}^0(\mathbb{Z}_p, \mathbf{o}_\mathbf{K}) \xrightarrow{\sim} \varprojlim \mathbf{o}_\mathbf{K}[\mathbb{Z}/p^n\mathbb{Z}] =: \mathbf{o}_\mathbf{K}[[\mathbb{Z}_p]], \quad \text{and}$$

- by the *Iwasawa isomorphism* that maps the generator 1 of $\mathbb{Z}_p$ to $1 + X$

$$\mathbf{o}_\mathbf{K}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathbf{o}_\mathbf{K}[[X]].$$

Mahler Basis

Theorem (Schikhof Duality)

Let $\binom{\cdot}{n}\colon \mathbb{Z}_p \to \mathbf{K}$ *with* $\binom{x}{n} := x(x-1)\cdots(x-n+1)/n!$. *Then*

$$\{\text{ all zero sequences over } \mathbf{K} \,\} \xrightarrow{\sim} \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$$
$$(a_n) \mapsto \sum a_n \binom{\cdot}{n}$$

Proof.

Because $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \{\text{ all bounded sequences over } \mathbf{K} \,\}$. □

Let $r \geq 0$. Denote

- $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) := \{$ all r-times differentiable $f\colon \mathbb{Z}_p \to \mathbf{K}\}$,
- $\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K}) := \{$ all continuous linear $\nu\colon \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}\}$,

and

$$d^r(\mathbb{N}, \mathbf{K}) := \{ \text{ all } \sum a_n X^n \text{ in } \mathbf{K}[[X]] \text{ with } \{|a_n|/n^r\} \text{ bounded } \}.$$

Theorem

We have $\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} d^r(\mathbb{N}, \mathbf{K})$ *as normed* $\mathbf{K}$*-vector spaces.*

Back to φ, Γ-modules

Let $n = 2$, that is,

$$\mathrm{V} = \mathbf{K} \oplus \mathbf{K}.$$

If $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \circlearrowright \mathrm{V}$ is "effective crystalline" then its φ, Γ-module D over $\mathscr{E}$ is

- base extended from

$$\text{a } \varphi, \Gamma\text{-module } \mathrm{N} \text{ over } d^0(\mathbb{N}, \mathbf{K}), \quad \text{and}$$

- and, for some $r, s \geq 0$,

$$\mathrm{N} \text{ is a } \varphi, \Gamma\text{-submodule of } d^r(\mathbb{N}, \mathbf{K}) \oplus d^s(\mathbb{N}, \mathbf{K}).$$

Matrix Action

Fourier transform $\rightsquigarrow$ the φ, Γ-module N is

- a module over $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K})$,
- a submodule of $\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K}) \oplus \mathscr{D}^s(\mathbb{Z}_p, \mathbf{K})$.

In particular

- $\mathbb{Z}_p \circlearrowleft \mathrm{N}$ by δ_x in $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K})$ for all x in $\mathbb{Z}_p$,
- $\mathbb{Z}_p^* \circlearrowleft \mathrm{N}$ by $\mathbb{Z}_p^* = \Gamma$ and $p^{\mathbb{N}} \circlearrowleft \mathrm{N}$ by $p = \varphi$

Thus,

$$\mathrm{M} := \begin{pmatrix} p^{\mathbb{N}}\mathbb{Z}_p^* & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix},$$

that is, $\mathrm{M}^+ = p^{\mathbb{N}}\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ acts on N.

$\mathrm{N} \rightsquigarrow \bar{\mathrm{N}}$ = submodule of $\mathscr{D}^r(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^s(\mathbb{Q}_p, \mathbf{K})$ over $\mathscr{D}^0(\mathbb{Q}_p, \mathbf{K})$
Then there is an action of

$$\begin{pmatrix} \mathbb{Q}_p^* & \mathbb{Q}_p \\ & \mathbb{Q}_p^* \end{pmatrix}$$

on $\bar{\mathrm{N}}$ which extends uniquely to one of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\bar{\mathrm{N}}$.
Dualizing gives the sought-for Banach space
$\bar{\mathrm{N}} \rightsquigarrow \mathrm{GL}_2(\mathbb{Q}_p) \circlearrowright \mathrm{B} :=$ subquotient of $\mathscr{C}^r(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{C}^s(\mathbb{Q}_p, \mathbf{K})$

1. p-adic Langlands program
2. From Characteristic 0 to p
3. Fourier Transform

Notes available at

`http://imj-prg.fr/~enno.nagel`.