PAINLEVÉ VI EQUATION

IN p–ADIC TIME

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Plan

- **Part I: INTRODUCTION.**

A. BUIUM's CALCULUS IN p–ADIC DIRECTION

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PAINLEVÉ VI

• **Painleve VI family.** **Equations of the type Painlevé VI form a four–parametric family. Denote parameters** $(\alpha, \beta, \gamma, \delta)$**: the corresponding equation for a function** $X(t)$ **looks as follows:**

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right].$$

• **Some physically relevant equations :**

(A) CASE $(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$. **Solutions in elliptic functions of this equation describe Bianchi IX space–times with SU(2)–symmetry. Here the independent variable** t **plays role of cosmological time: see [To], [Hi].**

Below we will show how to replace d/dt **by "derivation in** p**–adic direction".**

(B) CASE $(\alpha, \beta, \gamma, \delta) = (\frac{9}{2}, 0, 0, \frac{1}{2})$. **A specific solution of this equation describes "the mirror of** $\mathbf{P}^2$**" in a general context of Mirror Symmetry.**

BUIUM′S p – DIFFERENTIAL CALCULUS

(A) TABLE OF ANALOGIES

POWER SERIES	*p – ADICS*
$\sum a_i t^i \in F := K[[t]]$ *or* $K((t))$	$\sum \varepsilon_i p^i \in R := \mathbf{Z}_p^{un}$ *or* $\mathbf{Q}_p^{un}$
Field of constants: $a_i \in K$	**Monoid:** $\varepsilon_i \in \mu_\infty \cup \{0\}$ *(Teichmüller representatives)*
Derivation: d/dt	$\delta_p(*) := \frac{\Phi(*) - *^p}{p}$ $(\Phi := lift\ of\ Frobenius)$
Polynomial Diff. Operators: $D \in F[T_0, T_1, \ldots, T_n]$	**p-adic PDO:** $D_p \in \overline{R[T_0, T_1, \ldots, T_n]}$ **(p-adic completion!)**

Action of PDO: $f \mapsto D(f, f', \dots f^{(n)})$ **or** $D_p(f, \delta_p f, \dots, \delta_p^n f)$

APPLICATIONS

- **EXAMPLE 1:** p**–adic "logarithmic derivative", analog of**

$$\mathbf{G}_m(F) \to \mathbf{G}_a(F): \quad f \mapsto f'/f = \frac{d}{dt} \ln f,$$

is the differential character $\mathbf{G}_m(R) \to \mathbf{G}_a(R):$

$$a \mapsto \delta_p a \cdot a^{-p} - \frac{p}{2}(\delta_p a \cdot a^{-p})^2 + \frac{p^2}{3}(\delta_p a \cdot a^{-p})^3 - \dots$$

- **EXAMPLE 2: Quadratic reciprocity symbol:**

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \left(1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2k-2)!}{2^{2k-1}(k-1)!k!} (\delta_p a)^k a^{-pk}\right)$$

• **EXAMPLE 3:** p–**adic analog** μ_p **of the character** μ **defined upon sections** $P := (X(t), Y(t))$ **of the generic elliptic curve** $E = E(t): \ \ Y^2 = X(X-1)(X-t)$. **The section can be local and/or multivalued. Put**

$$\mu(P) = (4t(1-t)\frac{d^2}{dt^2} + 4(1-2t)\frac{d}{dt} - 1)\int_{\infty}^{P} \frac{dX}{Y}$$

It is a non–linear differential expression (additive differential character) in coordinates of P **such that** $\mu(P+Q) = \mu(P)+\mu(Q)$ **where** $P+Q$ **means addition of points of the generic elliptic curve** E**, with infinite section as zero.**

In particular, $\mu(Q) = 0$ **for points of finite order.**

This example is critically important for us, because it is directly related to Painlevé VI. Here are some details.

• **<u>Painleve VI revisited.</u> (A) In 1907, R. Fuchs has rewritten PVI in the form**

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_\infty^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} \qquad (*)$$

Here he enhanced $X := X(t)$ to $(X,Y) := (X(t),Y(t))$ treating the latter pair as a section $P := (X(t),Y(t))$ of the generic elliptic curve $E=E(t):\ Y^2=X(X-1)(X-t)$.

(B) The meaning of the right hand side of $(*)$ was clarified in my paper of 1996:

if we pass to the analytic picture replacing the algebraic family of curves $E(t)$ by the analytic one $E_\tau := \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau) \mapsto \tau \in H$, and denote by z a fixed coordinate on C, then (*) can be equivalently written in the form

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z\left(z + \frac{T_j}{2}, \tau\right) \qquad (**)$$

Here $(\alpha_0, \dots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$**,** $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)$**, and**

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right).$$

Moreover, we have

$$\wp_z(z, \tau)^2 = 4(\wp(z, \tau) - e_1(\tau))(\wp(z, \tau) - e_2(\tau))(\wp(z, \tau) - e_3(\tau))$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2}, \tau),$$

so that $e_1 + e_2 + e_3 = 0$**.**

p–ADIC DIFFERENTIAL EQUATIONS: BACKGROUND ACCORDING TO A. BUIUM

• $p-$ **derivations.** **Recall that, in the conventional commutative algebra, given a ring** A **and an** A**–module** N, *a derivation of* A *with values in* N **is any map** $\partial: A \to N$ **such that the map**

$$A \to A \times N : a \mapsto (a, \partial a)$$

is a ring homomorphism, where $A \times N$ **is endowed with the structure of commutative ring with componentwise addition, and multiplication** $(a, m) \cdot (b, n) := (ab, an + bm)$**. Notice that** $\{0\} \times N$ **is the ideal with square zero in** $A \times N$**.**

Similarly, in arithmetic geometry *a p–derivation of A with values in an A–algebra B, $f : A \to B$,* **is a map $\delta_p : A \to B$ such that the map**

$$A \to B \times B : a \mapsto (f(a), \delta_p(a))$$

is a ring homomorphism $A \to W_2(B)$ where $W_2(B)$ is the ring of *p–typical Witt vectors of length 2.*

Again, if $pB = \{0\}$, Witt vectors of the form $(0, b)$ form the ideal of square zero.

Explicitly, this means that $\delta_p(1) = 0$**, and**

$$\delta_p(x+y) = \delta_p(x) + \delta_p(y) + C_p(x,y),$$

$$\delta_p(xy) = f(x)^p \cdot \delta_p(y) + f(y)^p \cdot \delta_p(x) + p \cdot \delta_p(x) \cdot \delta_p(y),$$

where

$$C_p(X,Y) := \frac{X^p + Y^p - (X+Y)^p}{p} \in \mathbf{Z}[X,Y].$$

• $p-$ **derivations and lifts of Frobenius. In particular, this implies that for any** p–**derivation** $\delta_p : A \to B$ **the respective map** $\phi_p : A \to B$ **defined by**

$$\phi_p(a) := f(a)^p + p\delta_p(a)$$

is a ring homomorphism satisfying

$$\phi_p(x) \equiv f(x)^p \bmod p$$

that is, *"a lift of the Frobenius map applied to f"***.**

Conversely, having such a lift of Frobenius, we can uniquely reconstruct the respective derivation δ_p *under the condition that B has no p–torsion.*

• <u>Conventions</u>. **We will work with p–derivations $A \to A$ with respect to the identity map $A \to A$ and keep p fixed.**

Such a pair (A, δ) is called a *δ–ring*. **Morphisms of δ–rings are algebra morphisms compatible with their p–derivations.**

Moreover, our rings (and more generally, schemes) will be R–algebras where $R = W(k)$ (ring of infinite p–typical Witt vectors) is the completion of the maximal unramified extension of $\mathbf{Z}_p$, with residue field k := an algebraic closure of $\mathbf{Z}/p\mathbf{Z}$. By $\phi : R \to R$ we denote the automorphism acting as Frobenius $x \mapsto x^p$ on k, and by δ the respective p–derivation: $\delta(x) = (\phi(x) - x^p)/p$.

The R–algebra structure on a δ–ring is always assumed to be compatible with this p-derivation.

• **<u>Prolongation sequences and $p-$jet spaces.</u> In the classical situation, there exists an** *universal derivation*

$$d:\ A \to \Omega^1(A)$$

with values in the A–module of differentials.

For p–derivations, this might be replaced by the following construction (however, see the Remark below).

Let A be an R–algebra. *A prolongation sequence* **for A consists of a family of p–adically complete R–algebras $A^i, i \geq 0$, where $A^0 = A\widehat{\ }$ is the p–adic completion of A, and of maps $\varphi_i, \delta_i :$ $A^i \to A^{i+1}$ satisfying the following conditions:**

a) φ_i *are ring homomorphisms, each* δ_i *is a p–derivation with respect to* φ_i*, compatible with* δ *on* R*.*

b) $\delta_i \circ \varphi_{i-1} = \varphi_i \circ \delta_{i-1}$ *for all* $i \geq 1$*.*

Prolongation sequences form a *category.*

Its morphisms are ring homomorphisms $f_i : A^i \to B^i$ **commuting with** φ_i **and** δ_i**, and in its subcategory with fixed** A^0 *there exists an initial element, defined up to unique isomorphism* **(cf. [Bu05], Chapter 3). It can be called the** *universal prolongation sequence* **(for** A^0**).**

• **<u>The geometric language</u>. If $X = \mathrm{Spec}\, A$, the formal spectrum of the i–th ring A^i in the universal prolongation sequence is denoted $J^i(X)$ and called** *the i–th p–jet space of X.* **Conversely, $A^i = \mathcal{O}(J^i(X))$, the ring of global functions.**

The geometric morphisms (of formal schemes over $\mathbf{Z}$) corresponding to ϕ_i are denoted $\phi^i : J^i(X) \to J^0(X) =: X\hat{}$(formal p–adic completion of X).

This construction is compatible with localisation so that it can be applied to the non–necessarily affine schemes: cf. [Bu], Chapter 3.

• **Remark and warning. Classically, the map** $d: A \to \Omega^1(A)$ **extends to the universal map of** A **to the differential graded algebra** $\Omega^*(A)$**, and there is a superficial similarity of this map with the one, say of** A **to the inductive limit of its universal prolongation sequence in the** p**–adic arithmetics context.**

However, the classical differential acts in $\mathbf{Z}_2$*–graded supercommutative algebras* **and is an** *odd* **operator with square zero, whereas** δ_p **are even.**

The differential geometry of smooth schemes *in characteristic* $p > 0$ **suggests a perspective worth exploring.**

Namely, the sheaf of differential forms on such a scheme is endowed with the so called *Cartier operator* C**, which is dual to the Frobenius operator** $F : \partial \mapsto \partial^p$ **acting upon vector fields. This operator** C **is** F^{-1}**–linear.**

One could consider studying p**–adic lifts of the Cartier operator from the closed fibre of the relevant scheme to its** p**–adic completion, following the lead of [Bu].**

• **Flows**. **Let now** X **be a smooth affine scheme over** $R = W(k)$**. Each element of** $\mathcal{O}(J^r(X))$ **induces a function** $f : X(R) \mapsto R$**.**

Such functions are called *δ_p–functions of order r on X*, **and we may and will identify them with respective elements of** $\mathcal{O}(J^r(X))$**. For** $r = 0$, $\mathcal{O}(J^0(X)) = \mathcal{O}(X)\widehat{\ }$, **the** p**–adic completion of** $\mathcal{O}(X)$**.**

• **Definition**. *a) A system of arithmetic differential equations of order r on X is a subset $\mathcal{E}$ of $\mathcal{O}(J^r(X))$.*

b) A solution of $\mathcal{E}$ *is an* R*–point* $P \in X(R)$ *such that* $f(P) = 0$ *for all* $f \in \mathcal{E}$*. The set of solutions of* $\mathcal{E}$ *is denoted* $Sol(\mathcal{E}) \subset X(R)$*.*

c) A prime integral of $\mathcal{E}$ *is a function* $\mathcal{H} \in \mathcal{O}(X)\widehat{\ }$ *such that* $\delta(\mathcal{H}(P)) = 0$ *for all* $P \in Sol(\mathcal{E})$*.*

We will also denote by $Z^r(\mathcal{E})$ **the closed formal subscheme of** $J^r(X)$ **generated by** $\mathcal{E}$**.**

Now, let δ_X **be a** p**–derivation of** $\mathcal{O}(X)\widehat{\ }$**. From the universality of the jet sequence explained in 1.2, it follows that such derivations are in a bijection with the sections of the canonical morphism** $J^1(X) \to J^0(X)$**.**

• **Definition**. *The δ–flow associated to δ_X is the system of arithmetic differential equations of order 1 which is the ideal in $\mathcal{O}(J^1(X))$ generated by elements of the form $\delta f_i - \delta_X f_i$ where $f_i \in \mathcal{O}(X)$ generate $\mathcal{O}(X)$ as R–algebra.*

We use the word "flow" in this context in order to suggest that in our main applications *we consider the p–adic axis as an arithmetic version of the time axis.*

The derivation δ_X is completely determined by its δ–flow.

If δ_X **corresponds to the section** $s : J^0(X) \to J^1(X)$ **of** $J^1(X) \to J^0(X)$ **then** $Z(\mathcal{E}(\delta_X)) \subset J^1(X)$ **coincides with the image of this section.**

One easily checks that if $\mathcal{H} \in \mathcal{O}(X)\hat{}$ **is such that** $\delta_X \mathcal{H} = 0$ **then** $\mathcal{H}$ **is a prime integral for** $\mathcal{E}(\delta_X)$**. All of the above can be transposed to the case when** X **a** p**–formal scheme, locally a** p**–adic completion of a smooth scheme over** R**.**

In what follows we choose a smooth affine scheme Y **and apply the constructions discussed above to** $X := J^1(Y)$**. In this case one can define a special class of** δ**-flows on** $J^1(Y)$ **which will be called** *canonical* δ*-flows***.**

• **<u>Definition</u>.** *A canonical δ–flow is a δ–flow $\mathcal{E}(\delta_{J^1(Y)})$ with the property that the composition of $\delta_{J^1(Y)} : \mathcal{O}(J^1(Y)) \to \mathcal{O}(J^1(Y))$ with the pull back map $\mathcal{O}(Y) \to \mathcal{O}(J^1(Y))$ equals the universal p–derivation $\delta : \mathcal{O}(Y) \to \mathcal{O}(J^1(Y))$.*

Notice that in view of the universality property of p–jet spaces, one gets a natural closed embedding $\iota : J^2(Y) \to J^1(J^1(Y))$. This induces an injective map (which we view as an identification) from the set of sections of $J^2(Y) \to J^1(Y)$ to the set of sections of $J^1(J^1(Y)) \to J^1(Y)$. The sections of $J^2(Y) \to J^1(Y)$ are in a natural bijection with canonical δ–flows on $J^1(Y)$ whereas the sections of $J^1(J^1(Y)) \to J^1(Y)$ are in a bijection with (not necessarily canonical) δ–flows on $J^1(Y)$.

Finally, consider a system of arithmetic differential equations of order 2, $\mathcal{F} \subset \mathcal{O}(J^2(Y))$.

• **Definition.** *$\mathcal{F}$ defines a δ–flow on $J^1(Y)$ if the map $Z^2(\mathcal{F}) \to J^1(Y)$ is an isomorphism.*

In this case then $Z^2(\mathcal{F}) \to J^1(Y)$ defines a section of $J^2(Y) \to J^1(Y)$ and hence a canonical δ–flow $\mathcal{E}(\delta_{J^1(Y)})$ on $J^1(Y)$ such that $\iota(Z^2(\mathcal{F})) = Z^1(\mathcal{E}(\delta_{J^1(Y)}))$.

The differential algebra counterpart of the above definition yields the natural concept of flow on the (co)tangent space defining a second order differential equation.

p-ADIC PVI: LEFT HAND SIDE

- **Classical PVI : recollection :**

$$\frac{d^2z}{d\tau^2} = \frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j\wp_z(z+\frac{T_j}{2},\tau) \qquad (**)$$

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left(\frac{1}{(z-m\tau-n)^2} - \frac{1}{(m\tau+n)^2}\right).$$

Here $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1+\tau)$**, and**

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau)-e_1(\tau))(\wp(z,\tau)-e_2(\tau))(\wp(z,\tau)-e_3(\tau))$$

$$e_i(\tau) = \wp(\frac{T_i}{2},\tau),$$

so that $e_1 + e_2 + e_3 = 0$**.**

• $p-$ **adic version of** $\frac{\mathbf{d^2z}}{\mathbf{d\tau^2}}$: **differential additive characters.**

Let $p \geq 5$ **be a prime,** k **an algebraic closure of** $\mathbf{Z}/p\mathbf{Z}$**,** $R = W(k)$ **the ring of** p**–typical Witt vectors.**

Consider a smooth projective curve of genus one E **over** R**, with four marked and numbered** R**–sections** P_i**,** $i = 0, \ldots, 3$, **such that all divisors** $2(P_i - P_j)$ **are principal ones.**

Choosing P_0 **as zero section, we may represent** E **as the closure of the affine curve**

$$y^2 = 4x^3 + ax + b = 4(x - e_1)(x - e_2)(x - e_3),$$

with $e_i \in R$**, corresponding to** $P_i - P_0$**.**

Theorem (A. Buium). *Assume that E is not a canonical lift of its (good) reduction, that is, does not admit a lift of the Frobenius morphism of $E \otimes k$.*

Then E has a canonical p–differential character ψ_E of order 2 ([Bu], pp. 201 and 197), which corresponds to $(2\pi i)^2 d^2 z/d\tau^2$.

• **A $p-$adic version of the r. h. s. of$(**)$. Since $\wp_z(z,\tau)$ corresponds to the affine coordinate y, the most straightforward choice of $(**)_p$ is simply**

$$\psi_E(Q) = \sum_{j=0}^{3} \alpha_j s_j^*(y(Q)) \qquad (**)_p$$

where Q is a variable section of E/R, and $s_j : E_j \to E_j$ is the shift by P_j.

- **PVI$_\mathbf{p}$ and "constants" in the $p-$ adic direction.**

(A) In the classical setting (**), α_j must be absolute constants rather than functions of t. Directly imitating this condition, we have to postulate that in (**)$_p$, α_j must be p–unramified roots of unity or zero, i. e. δ_p–constants.

It is desirable to find a justification of such a requirement (or a version of it) in a more extended p–adic theory of Painlevé VI, e. g. tracing its source to the arithmetic analog of isomonodromy deformations.

(B) The relevance and non–triviality of the problem of "constants" in the p–adic differential equations context is also implicit in Buium's *exclusion of those elliptic curves that are canonical lifts of their reductions.*

A formal reason for this exclusion was the fact that for such E the basic differential character ψ has order 1 rather than 2, thus being outside the framework of Painlevé VI. But the analogy with functional case suggests that canonical lifts should be morally considered as *analogs of families with constant j–invariant* **in the functional case.**

This agrees also with J. Borger's philosophy that Frobenius lift(s) should be treated as descent data to an *"algebraic geometry below* $\mathrm{Spec}\,\mathbf{Z}$*"*.

Indeed, canonical lifts X **in such a geometry are endowed with an isomorphism** $X^{\phi} \to X^{(p)}$ **that can be seen as a categorification of the identity** $c^{\phi} = c^{p}$ **defining roots of unity.**

In the p**–adic case, however, if we decide to declare** j**–invariants of canonical lifts "constants" in some sense, this will require a revision of the latter notion.**

These invariants generally are *not* **roots of unity: cf. Finotti's papers in http://www.math.utk.edu/~finotti/ .**

A HAMILTONIAN VERSION OF PVI

• Classical case. The classical PVI equation $(**)$ can be represented as a Hamiltonian flow on the variable two–dimensional phase space (twisted cotangent spaces to a versal family of elliptic curves), with time–dependent Hamiltonian:

$$\frac{dz}{d\tau} = \frac{\partial \mathcal{H}}{\partial y}, \ \ \frac{dy}{d\tau} = -\frac{\partial \mathcal{H}}{\partial z},$$

where

$$\mathcal{H} := \mathcal{H}(\alpha_0, \ldots, \alpha_3) := \frac{y^2}{2} - \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp(z + \frac{T_j}{2}, \tau).$$

In more geometric terms, this means that solutions to the PVI become leaves of the null–foliation of the following closed two–form:

$$\omega = \omega(\alpha_0, \dots, \alpha_3) := 2\pi i(dy \wedge dz - d\mathcal{H} \wedge d\tau) =$$

$$= 2\pi i(dy \wedge dz - ydy \wedge d\tau) + \frac{1}{2\pi i}\sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau) dz \wedge d\tau.$$

The extra factor $2\pi i$ makes ω defined over $\mathbf{Q}[\alpha_i]$ on a natural algebraic model of (twisted) relative cotangent bundle to the respective versal family of elliptic curves.

In this expression, the summand $2\pi i\, dy \wedge dz$ is the canonical fibrewise symplectic form on the relative cotangent bundle. The terms involving $d\tau$ uniquely determine the differential of the (time dependent) Hamiltonian.

Moreover, ω is not just closed, but is a global differential: $\omega = d\nu$ where the form

$$\nu = \nu(\alpha_0, \dots, \alpha_3) := 2\pi i\,(y dz - \frac{1}{2} y^2 d\tau) + d\log\theta(z,\tau) + 2\pi i\, G_2(\tau) d\tau +$$

$$+ \frac{1}{2\pi i} \sum_{j=0}^{3} \alpha_j \wp(z + \frac{T_j}{2}, \tau) d\tau$$

also descends to an appropriate algebraic model, and the Hamiltonian $\mathcal{H}$ is again encoded in the $d\tau$–part of ν.

Here is a convenient way to represent this encoding:

$$\mathcal{H}(\alpha_0,\ldots\alpha_3)=i_{\partial_\tau}\left(\frac{y^2}{2}d\tau+\frac{1}{2\pi i}(\nu(0,\ldots,0)-\nu(\alpha_0,\ldots,\alpha_3))\right),$$

where $\partial_\tau := \frac{\partial}{\partial\tau}$**.**

Finally, the last summand is simply the additive differential character $2\pi i\dfrac{d^2z}{d\tau^2}$ **that is generally denoted** ψ **in the** p**–adic case.**

In the remainder of the talk, we try to transfer this Hamiltonian structure to p–adic PVI equations.

Our treatment is somewhat *ad hoc* **and tentative:**

we do not yet have an appropriate version of dp replacing $d\tau$ and generally do not know what are differential forms involving "differentials in the arithmetical direction".

• **Arithmetical case : preparation.** **Let** Y **be a formal affine scheme over** $R = W(k)$**. Modules of vertical differential forms on** Y **are defined as**

$$\Omega_Y = \lim \mathrm{inv}\, \Omega_{Y_n/R_n}$$

where $R_n = R/p^{n+1}R$**,** $Y_n = Y \otimes_R R_n$**.**

Let now $Z \subset J^n(Y)$ **be a closed formal subscheme defined by the ideal** $I_Z \subset \mathcal{O}(J^n(Y))$. **Put**

$$\Omega'_Z := \frac{\Omega_{J^n(Y)}}{\langle I_Z \Omega_{J^n(Y)}, dI_Z \rangle} \tag{4.6}$$

Given a system of arithmetic differential equations $\mathcal{F} \subset \mathcal{O}(J^r(Y))$**, denote by** $Z^r := Z^r(\mathcal{F})$ **the ideal generated by** $\mathcal{F}$**. For each** $s \leq r$**, there is a natural map** $\pi_{r,s} : Z^r \to J^s(Y)$.

Generally, the natural maps ϕ^* respect degrees of differential forms, one can define natural maps $\phi^*/p^i : \Omega^i_{J^{r-1}(Y)} \to \Omega^i_{J^r(Y)}$ and, for $f \in \mathcal{O}(J^2(Y))$, they induce maps which we will denote

$$\frac{\phi_Z^*}{p^i} : \Omega^i_{J^1(Y)} \to \Omega'^i_{Z^2(f)} \tag{4.7}$$

4.2.1. Definition. *We say that $\mathcal{F} \subset \mathcal{O}(J^r(Y))$ defines a generalised canonical δ–flow on $J^s(Y)$, if the induced map*

$$\pi^*_{r,s}\Omega_{J^s(Y)} \to \Omega'_{Z^r}$$

is injective, and its cokernel is annihilated by a power of p.

The cokernel here intuitively measures "how singular" $\mathcal{F}$ is on the closed fibre of Y.

4.2.2. Definition. *a) Let X be a smooth surface over R (or the p–adic completion of such a surface).*

A symplectic form on X is an invertible 2–form on X.

A contact form on X is an 1–form on X such that $d\nu$ is symplectic.

b) Let Y be a smooth curve over R. An 1–form on $X := J^1(Y)$ is called canonical, if $\nu = f\beta$, where $f \in \mathcal{O}(X)$ and β is an 1–form lifted from Y.

Notice that any closed canonical 1–form on $X = J^1(Y)$ is lifted from Y.

We now come to the main definition.

4.3. Definition. *Let Y be a smooth affine curve over R and let $f \in \mathcal{O}(J^2(Y))$ be a function defining a generalised canonical δ–flow on $J^1(Y)$.*

a) The respective generalized δ–flow is called Hamiltonian with respect to the symplectic form η on $J^1(Y)$, if $\phi_Z^*\eta = \mu \cdot \eta$ in $\Omega'^2_{Z^2(f)}$ for some $\mu \in pR$ called the eigenvalue.

b) Assume that moreover $\eta = d\nu$ for some canonical 1–form ν on $J^1(Y)$. Then we call

$$\epsilon := \frac{\phi_Z\nu - \mu\nu}{p} \in \Omega'_{Z^2(f)} \tag{4.8}$$

an Euler–Lagrange form.

We consider (4.8) as an (admittedly, half–baked) arithmetical analog of the expression $i_{\partial_\tau}(ydz - \mathcal{H}d\tau)$ (cf. (4.5)) in the same sense as the p–derivation

$$\delta_p(x) = \frac{\phi(x) - x^p}{p}$$

is an analog of ∂_τ.

Now we pass to the arithmetical PVI. Let again E be an elliptic curve over R that does not admit a lift of Frobenius and let $\psi \in \mathcal{O}(J^2(E))$ be the canonical δ-character of order 2 attached to an invertible 1–form ω on E. Consider the symplectic form $\eta = \omega^{(0)} \wedge \omega^{(1)}$ on $J^1(E)$: cf. (3.4). Let $Y \subset E$ be an affine open set and let $r \in \mathcal{O}(Y)$. Assume in addition that Y has an étale coordinate. (The basic example is E with sections of the second order deleted).

Denoting such an étale coordinate by T, put $\mathcal{A}_2 = K[[T, T']]$, $\mathcal{A}_3 = K[[T, T', T'']]$, where K is the quotient ring of R.

4.4. Proposition. *The following assertions hold:*

1) The function $f = \psi - \phi(r)$ defines a generalised canonical δ–flow on $J^1(Y)$ which is Hamiltonian with respect to η.

2) There exists a canonical 1*-form* ν *on* X *such that* $d\nu = \eta$*; in particular the symplectic form* η *is exact and if* $\epsilon := \frac{1}{p}(\phi_Z^*\nu - \mu\nu)$ *is the Euler-Lagrange form then* $p\epsilon$ *is closed.*

3) Let ϵ *be Euler-Lagrange form and* $f = \psi - \phi(r)$*. Then we have the following equality in* $\Omega_{\mathcal{A}_3}$*:*

$$\epsilon = f\omega^{(1)} - \frac{1}{p}(\phi^* - \mu)\nu.$$

4) Let $r_1, r_2 \in \mathcal{O}(Y)$ *be such that* $r_2 - r_1 = \partial s$*, for some* $s \in \mathcal{O}(Y)$*, where* ∂ *is the derivation on* E *dual to* ω*. (This holds for two right hand sides of any two PVI equations). Consider the equations* $\psi - \phi(r_1)$ *and* $\psi - \phi(r_2)$ *respectively. Then there exists a canonical* 1*–form* ν *on* $J^1(Y)$ *such that* $d\nu = \eta$ *and such that, if* ϵ_1, ϵ_2 *are the corresponding Euler–Lagrange forms, then :*

$$\epsilon_1 - \epsilon_2 = \frac{1}{p} d\phi(s) \in \Omega_{\mathcal{A}_2}.$$

Remark. We will deduce from this Proposition below (cf. Corollary 5.3.3) that in fact p**–adic PVI in the form (3.6)**

defines a generalised canonical δ–flow. However, it does not define a δ–flow in the sense of Definition 1.3.4. This motivated our Definition 4.2.1.

Proof. **From (3.4), we get the following equality in $\Omega^2_{J^2(Y)}$:**

$$\frac{\phi^*\eta}{p^2} = \frac{\phi^*\omega^{(0)}}{p} \wedge \frac{\phi^*\omega^{(1)}}{p} = \omega^{(1)} \wedge \omega^{(2)}.$$

Recall the formula (3.9b):

$$df = p\omega^{(2)} + (\lambda_1 - p\phi(\partial r))\omega^{(1)} + \lambda_0\omega^{(0)}$$

in $\Omega_{J^2(E)}$ where $\lambda_1 \in R$, $\lambda_0 \in R^\times$. Hence, if we keep notation $\omega^{(0)}, \omega^{(1)}$ also for the images of the respective forms in $\Omega'_{Z^2(f)}$, in view of $\omega^{(1)} \wedge df = 0$ in $\Omega'_{Z^2(f)}$, we have the following equality in $\Omega'^2_{Z^2(f)}$:

$$\phi_Z^*\eta = -p \cdot \omega^{(1)} \wedge ((\lambda_1 - p\phi(dr))\omega^{(1)} + \lambda_0\omega^{(0)}) = p\lambda_0\eta,$$

This completes the proof of 1).

Write now $\omega = dL = \frac{dL}{dT}dT$ **where** $L = L(T) \in TK[[T]]$**. (For instance, we can take** L **to be the formal logarithm of** E**). Then**

$$\mathcal{A}_3 = K[[T, \phi(T), \phi^2(T)]] = K[[L, \phi(L), \phi^2(L)]].$$

So the image of ψ **in** $\mathcal{A}_3$ **is**

$$\psi = \frac{1}{p}(\phi^2(L) + \lambda_1 \phi(L) + p\lambda_0 L) + \lambda_{-1}$$

for some $\lambda_{-1} \in R$**. Therefore the maps**

$$\Omega_{\mathcal{A}_2} \to \frac{\Omega_{\mathcal{A}_3}}{\langle f\Omega_{\mathcal{A}_3}, df\rangle}, \qquad \Omega^2_{\mathcal{A}_2} \to \frac{\Omega^2_{\mathcal{A}_3}}{\langle f\Omega^2_{\mathcal{A}_3}, df \wedge \Omega_{\mathcal{A}_3}\rangle}$$

are isomorphisms and so we have induced Frobenii maps $\phi_f^* : \Omega_{\mathcal{A}_2} \to \Omega_{\mathcal{A}_2}$ **and** $\phi_f^* : \Omega^2_{\mathcal{A}_2} \to \Omega^2_{\mathcal{A}_2}$**. Since** T **is étale the lift of Frobenius** $T \mapsto T^p$ **on** $\mathbf{A}^1$ **extends to a lift of Frobenius** $\phi_0 : \widehat{Y} \to \widehat{Y}$ **of the** p**–adic completion of** Y**. Also the derivation** $\frac{d}{dT}$ **on** $R[T]$ **extends to a derivation still denoted by** $\frac{d}{dT}$ **on**

$\mathcal{O}(\widehat{Y})$. We claim that

$$\frac{\phi(L)-\phi_0(L)}{p},$$

which a priori is an element of $\mathcal{A}_2$, actually belongs to $\mathcal{O}(J^1(Y))$. Indeed we have the following expansion in $\mathcal{A}_2$:

$$\frac{\phi(L)-\phi_0(L)}{p} = \frac{L^{(\phi)}(T^p+pT')-L^{(\phi)}(T^p)}{p}$$
$$=\sum_{i=1}^{\infty}\frac{p^{i-1}}{i!}\frac{d^iL^{(\phi)}}{dT^i}(T^p)(T')^i=\sum_{i=1}^{\infty}\frac{p^{i-1}}{i!}\phi_0\left(\frac{d^iL}{dT^i}\right)(T')^i$$
$$=\sum_{i=1}^{\infty}\frac{p^{i-1}}{i!}\phi_0\left(\left(\frac{d}{dT}\right)^{i-1}\left(\frac{\omega}{dT}\right)\right)(T')^i\in\mathcal{O}(J^1(Y)),$$

where the superscript (ϕ) means twisting coefficients by ϕ. The latter inclusion follows because $T'=\delta T\in\mathcal{O}(J^1(Y))$, $\frac{\omega}{dT}\in\mathcal{O}(Y)\subset\mathcal{O}(\widehat{Y})$, and the latter is stable under $\frac{d}{dT}$ and ϕ_0. Now set

$$\nu:=-\frac{\phi(L)-\phi_0(L)}{p}\,\omega\in\mathcal{O}(J^1(Y))\,\omega\in\Omega_{J^1(Y)}.$$

Then

$$d\nu = -d\left(\frac{\phi(L) - \phi_0(L)}{p}\right) \wedge \omega$$

$$= -d\left(\frac{\phi(L)}{p}\right) \wedge \omega + d\left(\frac{\phi_0(L)}{p}\right) \wedge \omega$$

$$= -\omega^{(1)} \wedge \omega^{(0)} = \eta.$$

This completes the proof of 2).

Next for $f = \psi - \phi(r)$ **we have the following computation in** $\Omega_{\mathcal{A}_3}$**:**

$$p\epsilon = \phi_f^* \nu - \mu\nu$$

$$= -\phi_f^*\left(\frac{\phi(L) - \phi_0(L)}{p}\omega\right) + \mu\frac{\phi(L) - \phi_0(L)}{p}\omega$$

$$= -\phi_f\phi(L)\omega^{(1)} + \phi\phi_0(L)\omega^{(1)} + \mu\frac{\phi(L) - \phi_0(L)}{p}\omega$$

$$= (\lambda_1\phi(L) + p\lambda_0 L + \phi\phi_0(L) - p\phi(r) + p\lambda_{-1})\omega^{(1)} + \mu\frac{\phi(L) - \phi_0(L)}{p}\omega^{(0)}$$

$$= (\lambda_1\phi(L) + p\lambda_0 L + \phi\phi_0(L) - p\phi(r) + p\lambda_{-1})\omega^{(1)}$$
$$+\phi^*\left(\frac{\phi(L)-\phi_0(L)}{p}\omega^{(0)}\right) - (\phi^* - \mu)\nu$$
$$= \left(\phi^2(L) + \lambda_1\phi(L) + p\lambda_0 L - p\phi(r) + p\lambda_{-1}\right)\omega^{(1)} - (\phi^* - \mu)\nu$$
$$= pf\omega^{(1)} - (\phi^* - \mu)\nu.$$

This ends the proof of assertion 3). Assertion 4) follows from the fact that

$$\epsilon_1 - \epsilon_2 = \phi(\partial s)\omega^{(1)} = \frac{1}{p}d(\phi(s)).$$

Remarks. a) **Some of our arguments above break down if** $\psi - \phi(r)$ **is replaced by** $\psi - r$**. This is our main motivation for studying** $\psi - \phi(r)$**.**

b) **Assertion 3) implies that if** $[\epsilon]_2, [f\omega^{(1)}]_2 \in \frac{\Omega_{\mathcal{A}_3}}{(\phi^*-\mu)\Omega_{\mathcal{A}_2}}$ **are the images of** $\epsilon, f\omega^{(1)}$ **then**

$$[\epsilon]_2 = [f\omega^{(1)}]_2.$$

This justifies our suggestion that f is the "Euler–Lagrange equation attached to our Hamiltonian data". Assertion 4) says that the Euler–Lagrange forms of various PVI equations differ by exact forms.

c) **Finally, we could treat in this way also the multicomponent version of PVI and the degeneration PV as they are described in [Ta].**

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THANK YOU FOR YOUR ATTENTION!