Dynamics of rational maps on the projective line of the field of p-adic numbers

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Introduction

I. p-adic dynamical systems

A dynamical system is a couple (X, T) where $T : X \to X$ is a transformation on the space X. We call (X, T) a p-adic dynamical system if X is a p-adic space.

The beginning :
Oselies-Zieschang 1975 : automorphisms of $\mathbb{Z}_p$
Herman-Yoccoz 1983 : complex p-adic dynamical systems
Volovich 1987 : p-adic string theory

Example : $(\mathbb{Z}_p, f)$ with $f \in \mathbb{Z}_p[x]$ being a polynomial. It is 1-**Lipschitz** and then equicontinuous. The system (X, T) is equicontinuous if

$$\forall \epsilon > 0, \exists \delta > 0 \ \text{ s. t. } \ d(T^n x, T^n y) < \epsilon \ (\forall n \geq 1, \forall d(x, y) < \delta).$$

Theorem

Let X be a compact metric space and $T : X \to X$ be an *equicontinuous transformation*. Then the following statements are equivalent :
(1) T is **minimal** (every orbit is dense).
(2) T is **uniquely ergodic** (there is a unique invariant measure).
(3) T is **ergodic** for any/some invariant measure with X as its support.

II. 1-Lipschitz continuous dynamics on $\mathbb{Z}_p$

For 1-Lipschitz continuous maps $f : \mathbb{Z}_p \to \mathbb{Z}_p$, the dynamical systems $(\mathbb{Z}_p, f)$ are extensively studied. For example :

- Polynomials :
  Coelho-Parry 2001 : ax and distribution of Fibonacci numbers
  Gundlach-Khrennikov-Lindahl 2001 : ergodicity of x^n on cycles.
  A. Fan-Li-Yao-Zhou 2007 : minimal decomposition of $ax + b$.
  Diarra-Sylla 2014 : periodic orbits of Chebyshev polynomials.
  S. Fan-Liao 2015 : minimal decomposition of x^2.
- Mahler Series
  Anashin 1994, 1995, 1998, 2002.
- van der Put Series
  Yurova 2010 ; Anashin-Khrennikov-Yurova 2011, 2012, 2014 ; Khrennikov-Yurova 2011 ; Jeong 2012.
- T-functions
  Anashin-Khrennikov-Yurova 2014.

Polynomials on $\mathbb{Z}_p$

I. Polynomial dynamical systems on $\mathbb{Z}_p$

- Let $f \in \mathbb{Z}_p[x]$ be a polynomial with coefficients in $\mathbb{Z}_p$.
- Polynomial dynamical systems : $f : \mathbb{Z}_p \to \mathbb{Z}_p$, noted as $(\mathbb{Z}_p, f)$.

Theorem (Ai-Hua Fan, L ; Adv. Math. 2011) minimal decomposition

Let $f \in \mathbb{Z}_p[x]$ with $\deg f \geq 2$. The space $\mathbb{Z}_p$ can be decomposed into three parts :

$$\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where

- $\mathcal{P}$ is the finite set consisting of all periodic orbits ;
- $\mathcal{M} := \sqcup_{i \in I} \mathcal{M}_i$ (I finite or countable)
  $\to$ $\mathcal{M}_i$: finite union of balls,
  $\to$ $f : \mathcal{M}_i \to \mathcal{M}_i$ is minimal ;
- $\mathcal{B}$ is attracted into $\mathcal{P} \sqcup \mathcal{M}$.

II. Dynamics for each minimal part

Given a positive integer sequence $(p_s)_{s\geq 0}$ such that $p_s | p_{s+1}$.
Profinite groupe : $\mathbb{Z}_{(p_s)} := \lim_{\leftarrow} \mathbb{Z}/p_s\mathbb{Z}$.
Odometer : The transformation $\tau : x \mapsto x+1$ on $\mathbb{Z}_{(p_s)}$.

Theorem (Chabert–A. Fan–Fares 2009)

Let E be a compact set in $\mathbb{Z}_p$ and $f : E \to E$ a 1-lipschitzian transformation. If the dynamical system (E, f) is minimal, then

- (E, f) is conjugate to the odometer $(\mathbb{Z}_{(p_s)}, \tau)$ where (p_s) is determined by the structure of E.

Theorem (Fan–L 2011 : Minimal components of polynomials)

Let $f \in \mathbb{Z}_p[X]$ be a polynomial and $O \subset \mathbb{Z}_p$ a clopen set, $f(O) \subset O$. Suppose $f : O \to O$ is minimal.

- If $p \geq 3$, then $(O, f|_O)$ is conjugate to the odometer $(\mathbb{Z}_{(p_s)}, \tau)$ where
$$(p_s)_{s\geq 0} = (k, kd, kdp, kdp^2, \dots) \quad (1 \leq k \leq p, d|(p-1)).$$
- If $p = 2$, then $(O, f|_O)$ is conjugate to $(\mathbb{Z}_2, x+1)$.

III. Minimality on the whole space $\mathbb{Z}_p$

Theorem (Larin 2002), General polynomials, only for $p = 2$

Let $p = 2$ and let $f(x) = \sum a_k x^k \in \mathbb{Z}_2[X]$ be a polynomial. Then $(\mathbb{Z}_p, f)$ is minimal iff

$$\begin{aligned} a_0 &\equiv 1 \pmod 2, \\ a_1 &\equiv 1 \pmod 2, \\ 2a_2 &\equiv a_3 + a_5 + \cdots \pmod 4, \\ a_2 + a_1 - 1 &\equiv a_4 + a_6 + \cdots \pmod 4. \end{aligned}$$

General polynomials for $p = 3$: **Durand-Paccaut 2009**.

Quadratic polynomials for all p : **Larin 2002 + Knuth 1969**.

IV. Minimal decomposition of affine polynomials on $\mathbb{Z}_p$

Let $T_{a,b}x = ax + b \quad (a, b \in \mathbb{Z}_p)$. Denote

$$\mathbb{U} = \{z \in \mathbb{Z}_p : |z| = 1\}, \quad \mathbb{V} = \{z \in \mathbb{U} : \exists m \geq 1, \text{s.t. } z^m = 1\}.$$

Easy cases :

1. $a \in \mathbb{Z}_p \setminus \mathbb{U} \quad \Rightarrow$ one attracting fixed point $b/(1-a)$.
2. $a = 1, b = 0 \ \Rightarrow$ every point is fixed.
3. $a \in \mathbb{V} \setminus \{1\} \ \Rightarrow$ every point is on a ℓ-periodic orbit, with ℓ the smallest integer $\geqslant 1$ such that $a^\ell = 1$.

Theorem (AH. Fan, MT. Li, JY. Yao, D. Zhou 2007) Case $p \geq 3$:

4. $a \in (\mathbb{U} \setminus \mathbb{V}) \cup \{1\},\ v_p(b) < v_p(1-a) \ \Rightarrow p^{v_p(b)}$ minimal parts.
5. $a \in \mathbb{U} \setminus \mathbb{V},\ v_p(b) \geq v_p(1-a) \Rightarrow (\mathbb{Z}_p, T_{a,b})$ is conjugate to $(\mathbb{Z}_p, ax)$.

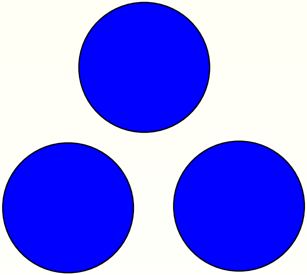
   Decomposition : $\mathbb{Z}_p = \{0\} \sqcup \sqcup_{n \geq 1} p^n \mathbb{U}$.

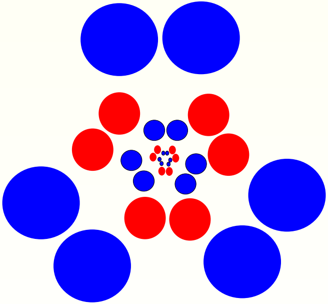
   (1) One fixed point $\{0\}$.

   (2) All $(p^n\mathbb{U}, ax)(n \geq 0)$ are conjugate to $(\mathbb{U}, ax)$.

   For $(\mathbb{U}, T_{a,0})$: $p^{v_p(a^\ell - 1)}(p-1)/\ell$ minimal parts, with ℓ the smallest integer $\geqslant 1$ such that $a^\ell \equiv 1 (\mathrm{mod}\ p)$.

Two typical decompositions of $\mathbb{Z}_p$





V. One application in Number Theory

Proposition (Fan-Li-Yao-Zhou 2007)

Let $k \geqslant 1$ be an integer, and let a, b, c be three integers in $\mathbb{Z}$ coprime with $p \geqslant 2$. Let s_k be the least integer $\geqslant 1$ such that $a^{s_k} \equiv 1 \pmod{p^k}$.

(a) If $b \not\equiv a^j c \pmod{p^k}$ for all integers j $(0 \leqslant j < s_k)$, then $p^k \nmid (a^n c - b)$, for any integer $n \geqslant 0$.

(b) If $b \equiv a^j c \pmod{p^k}$ for some integer j $(0 \leqslant j < s_k)$, then we have

$$\lim_{N \to +\infty} \frac{1}{N} \mathrm{Card}\{1 \leqslant n < N : p^k \mid (a^n c - b)\} = \frac{1}{s_k}.$$

Remark : Consider $T : x \mapsto ax$. Then

$$p^k \mid (a^n c - b) \Leftrightarrow |T^n(c) - b|_p \leq p^{-k} \Leftrightarrow T^n(c) \in \overline{B}(b, p^{-k}).$$

Coelho and Parry 2001 : Ergodicity of p-adic multiplications and the distribution of Fibonacci numbers.

Rational maps of degree 1 on $\mathbb{P}^1(\mathbb{Q}_p)$

0. Rational maps on $\mathbb{Q}_p$

Mukhamedov and Rozikov 2004 : $\frac{x+a}{bx+c}$.

Khamraev and Mukhamedov 2006 : $\frac{ax^2}{bx+1}$.

Dragovich, Khrennikov and Mihajlović 2007 : rational maps of degree 1 on the adelic space.

Albeverio, Rozikov and Sattarov 2013 : $(2,1)$-rational maps on the field of p-adic complex numbers.

Sattarov 2015 : $(3,2)$-rational maps on the field of p-adic complex numbers.

I. Projective line over $\mathbb{Q}_p$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}_p^2 \setminus \{(0,0)\}$, we say that $(x_1, y_1) \sim (x_2, y_2)$ if $\exists \lambda \in \mathbb{Q}_p^*$ s.t.

$$x_1 = \lambda x_2 \text{ and } y_1 = \lambda y_2.$$

Projective line over $\mathbb{Q}_p$:

$$\mathbb{P}^1(\mathbb{Q}_p) := (\mathbb{Q}_p^2 \setminus \{(0,0)\}) / \sim$$

Spherical metric : for $P = [x_1, y_1], Q = [x_2, y_2] \in \mathbb{P}^1(\mathbb{Q}_p)$, define

$$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}.$$

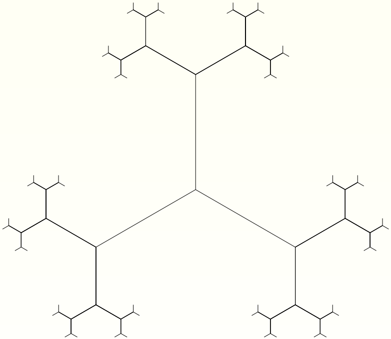
Viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

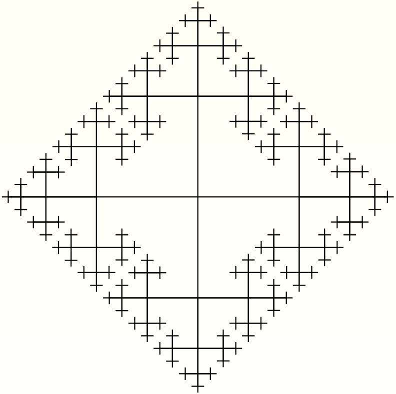
$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \qquad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1\,; \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$

Geometric representations of $\mathbb{P}^1(\mathbb{Q}_2)$ and $\mathbb{P}^1(\mathbb{Q}_3)$





II. Homographic maps

Let

$$\phi(x) = \frac{ax+b}{cx+d} \quad \text{with } a,b,c,d \in \mathbb{Q}_p,\ ad-bc \neq 0,$$

which induces an 1-to-1 map $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$.

The dynamics of ϕ depends on its fixed points which are the solution of

$$\frac{ax+b}{cx+d} = x \Leftrightarrow cx^2 + (d-a)x - b = 0.$$

Discriminant : $\Delta = (d-a)^2 + 4bc$.

- If $\Delta = 0$, then ϕ has only **one fixed point** x_0 in $\mathbb{Q}_p$ and $\phi(x)$ is conjugate to a translation $\psi(x) = x + \alpha$ for some $\alpha \in \mathbb{Q}_p$ by $g(x) = \frac{1}{x-x_0}$.
- If $\Delta \neq 0$ and $\sqrt{\Delta} \in \mathbb{Q}_p$, then ϕ has **two fixed points** $x_1, x_2 \in \mathbb{Q}_p$ and ϕ is conjugate to a multiplication $x \mapsto \beta x$ for some $\beta \in \mathbb{Q}_p$ by $g(x) = \frac{x-x_2}{x-x_1}$.
- If $\Delta \neq 0$ and $\sqrt{\Delta} \notin \mathbb{Q}_p$, then ϕ has **no fixed point** in $\mathbb{Q}_p$. But ϕ has two fixed points $x_1, x_2 \in \mathbb{Q}_p(\sqrt{\Delta})$. So we will study the dynamics of ϕ on $\mathbb{P}^1(\mathbb{Q}_p(\sqrt{\Delta}))$ then its restriction on $\mathbb{P}^1(\mathbb{Q}_p)$.

III. Minimal decomposition (ϕ admits no fixed point)

Theorem (AH. Fan, SL. Fan, L, YF. Wang ; Adv. Math. 2014)

Suppose that ϕ has no fixed points in $\mathbb{P}^1(\mathbb{Q}_p)$ and $\phi^n \neq id$ for each integer $n > 0$. Then

1. *the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as a finite number of minimal subsystems ;*
2. *these minimal subsystems are topologically conjugate to each other ;*
3. *the number of minimal subsystems is determined by the number*

$$\lambda := \frac{(a+d)+\sqrt{\Delta}}{(a+d)-\sqrt{\Delta}}.$$

Denote

- $K = \mathbb{Q}_p(\sqrt{\Delta})$ be the quadratic extension of $\mathbb{Q}_p$ generated by $\sqrt{\Delta}$.
- π be an uniformizer of K : $v_p(\pi) = 1/e$, where e is the ramification index of the extension. Define $v_\pi(x) := e \cdot v_p(x)$ for $x \in K$.
- $\mathbb{K}$ be the residue field of K.
- ℓ be the order in the group $\mathbb{K}^*$ of λ.

IV. The case $p \geq 3$

Theorem (FFLW, $K = \mathbb{Q}_p(\sqrt{N_p})$ is unramified ($e = 1$))

The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ *is decomposed as* $((p+1)p^{v_p(\lambda^\ell - 1) - 1})/\ell$ *minimal subsystems. Each subsystem is topologically conjugate to the adding machine on an odometer* $\mathbb{Z}_{(p_s)}$ *with* $(p_s) = (\ell, \ell p, \ell p^2, \cdots)$.

Theorem (FFLW, $K = \mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{pN_p})$ is ramified ($e > 1$))

(1) *If* $|a + d|_p > |\sqrt{\Delta}|_p$, *then* $\lambda = 1 \pmod{\pi}$. *The dynamics* $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ *is decomposed as* $2p^{(v_\pi(\lambda^p - 1) - 3)/2}$ *minimal subsystems. Moreover, each minimal subsystem is conjugate to the adding machine on the odometer* $\mathbb{Z}_{(p_s)}$ *with* $(p_s) = (1, p, p^2, \cdots)$.

(2) *If* $|a + d|_p < |\sqrt{\Delta}|_p$, *then* $\lambda = -1 \pmod{\pi}$. *The dynamics* $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ *is decomposed as* $p^{(v_\pi(\lambda^p + 1) - 3)/2}$ *minimal subsystems. Moreover, each minimal subsystem is conjugate to the adding machine on the odometer* $\mathbb{Z}_{(p_s)}$ *with* $(p_s) = (2, 2p, 2p^2, \cdots)$.

V. Minimal (ergodic) conditions

Corollary (FFLW, case $p \geq 3$)

The system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if one of the following conditions satisfied

(1) $K = \mathbb{Q}_p(\sqrt{\Delta})$ *is unramified,* $\ell = p + 1$ *and* $v_p(\lambda^\ell - 1) = 1$,

(2) $K = \mathbb{Q}_p(\sqrt{\Delta})$ *is ramified and* $v_\pi(\lambda^p + 1) = 3$.

Corollary (FFLW, case $p = 2$)

The system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is minimal if and only if one of the following conditions satisfied

(1) $K = \mathbb{Q}_2(\sqrt{\Delta}) = \mathbb{Q}_2(\sqrt{-3})$, $\ell = 3$ *and* $v_2(\lambda^{2\ell} - 1) = 2$,

(2) $K = \mathbb{Q}_2(\sqrt{\Delta}) = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{3})$, $|a + b|_2 = |\sqrt{\Delta}|_2$ *and* $v_\pi(\lambda^2 + 1) = 2$.

Rational maps with good reduction

I. Rational maps with good reduction

Let $\widetilde{\cdot} : \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$ be the reduction modulo p defined by $a \mapsto \widetilde{a} \equiv a \mod p$.

The reduction of a polynomial $f(z) = \sum_{i=0}^n a_i z^i \in \mathbb{Z}_p[z]$ is

$$\widetilde{f}(z) = \sum_{i=0}^{n} \widetilde{a}_i z^i.$$

A rational map $\phi(z) \in \mathbb{Q}_p(z)$ can be written as a quotient of polynomials $f(z), g(z) \in \mathbb{Z}_p[z]$ having no common factors, such that **at least one coefficient of f or g has absolute value** 1. The reduction of ϕ is

$$\widetilde{\phi}(z) = \frac{\widetilde{f}(z)}{\widetilde{g}(z)} \in \mathbb{F}_p(z).$$

If $\deg \widetilde{\phi} = \deg \phi$, we say ϕ has **good reduction**, and if $\deg \widetilde{\phi} < \deg \phi$, we say ϕ has **bad reduction**.

Theorem (Silverman's book : The arithmetic of dynamical systems)

A rational map ϕ has good reduction if and only if it is 1-Lipschitz continuous with respect to the spherical metric.

II. Minimal criterion for good reductive rational maps

If ϕ has good reduction, then $\widetilde{\phi}$ induces a map from $\mathbb{P}^1(\mathbb{F}_p)$ to itself, where $\mathbb{P}^1(\mathbb{F}_p)$ is the projective line over $\mathbb{F}_p$

Theorem (Fan-Fan-L-Wang, preprint)

Let $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$ be a rational map with good reduction and $\deg \phi \geq 2$. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the following conditions are satisfied

(1) *The reduction $\widetilde{\phi}$ is transitive on $\mathbb{P}^1(\mathbb{F}_p)$.*

(2) $(\phi^{p+1})'(0) = 1 \pmod{p}$.

(3) *For $p = 2$ or 3, $|\phi^{p+1}(0) - 0|_p = 1/p$ and $|\phi^{(p+1)p}(0) - 0|_p = 1/p^2$. For $p \geq 5$, $|\phi^{p+1}(0) - 0|_p = 1/p$.*

III. Min. decomp. for rational maps with good reduction

Theorem (Fan-Fan-L-Wang, preprint)

Let $\phi \in \mathbb{Q}_p(z)$ be a rational map with good reduction and $\deg \phi \geq 2$. We have the following decomposition

$$\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where

- $\mathcal{P}$ *is the finite set consisting of all periodic orbits ;*
- $\mathcal{M} := \sqcup_{i \in I} \mathcal{M}_i$ *(I finite or countable)*
  $\rightarrow$ $\mathcal{M}_i$ *: finite union of balls,*
  $\rightarrow$ $\phi : \mathcal{M}_i \to \mathcal{M}_i$ *is minimal ;*
- $\mathcal{B}$ *is attracted into* $\mathcal{P} \sqcup \mathcal{M}$.

Further if $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ is a minimal clopen invariant set of ϕ, then $\phi : E \to E$ is conjugate to the adding machine on an odometer $\mathbb{Z}_{(p_s)}$, where

$$(p_s) = (k, kd, kdp, kdp^2, \cdots)$$

with integers k and d such that $1 \leq k \leq p+1$ and $d \mid (p-1)$.

IV. Minimal (ergodic) conditions for $p = 2$

Theorem (Fan-Fan-L-Wang, preprint)

Each map in $\mathbb{Q}_2(z)$ is conjugate to a map of the form

$$\phi(z) = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n}{b_1 z + \cdots + b_{n-1} z^{n-1} + z^n}$$

with $n \geq 2$ and $a_i, b_j \in \mathbb{Q}_2$. Let $a_n = b_n = 1$ and set $A_\phi := \sum_{i\geq 0} a_i$, $B_\phi := \sum_{j\geq 1} b_j$, $A_{\phi,1} := \sum_{i\geq 0} a_{2i+1}$, $A_{\phi,2} := \sum_{i\geq 0} a_{4i+1}$ and $A_{\phi,3} := \sum_{i\geq 0} a_{4i+3}$. Then ϕ has good reduction and $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is minimal if and only if

$$\begin{cases} a_i, b_j \in \mathbb{Z}_2, & \text{for } 0 \leq i \leq n-1 \text{ and } 1 \leq j \leq n-1, \\ a_0 \equiv 1 \mod 2, \quad b_1 \equiv 1 \mod 2, \\ A_\phi \equiv 2 \mod 4, \quad A_{\phi,1} \equiv 1 \mod 2, \quad B_\phi \equiv 1 \mod 2, \\ a_{n-1} - b_{n-1} \equiv 1 \mod 2, \\ a_0 b_1 (a_{n-1} - b_{n-1})(A_{\phi,2} - A_{\phi,3}) B_\phi \\ +2(b_2 - a_1 + a_{n-2} - b_{n-2} + b_{n-1} + A_{\phi,3}) \equiv 1 \mod 4. \end{cases}$$

V. Corollaries

Corollary (Fan-Fan-L-Wang, preprint)

Let $\phi \in \mathbb{Q}_2(z)$ be a rational map of degree 2 or 3 having good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is not minimal.

Corollary (Fan-Fan-L-Wang, preprint)

We can find all rational maps of order 4 with good reduction which are minimal on $\mathbb{P}^1(\mathbb{Q}_2)$.

Example 1 : Let $p = 3$ and $\phi(z) = -\frac{2z^2+2z+1}{z^3-3z^2+z+1}$. The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal.

Example 2 : Let $p = 3$ and $\phi(z) = \frac{2z+3}{(z-1)(z-2)}$. The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal and we decompose $\mathbb{P}^1(\mathbb{Q}_p)$ as

$$\mathbb{P}^1(\mathbb{Q}_p) = B_1(0) \bigsqcup (\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)),$$

where $B_1(0)$ is a minimal component of ϕ and the points in $\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)$ are attracted to $B_1(0)$.

VI. Ideas and methods

- **Anashin 1994, Chabert–A. Fan–Fares 2009**, Local-Global Lemma :

  Let $X \subset \mathbb{Z}_p$ be a compact set.
  $f : X \to X$ is minimal $\Leftrightarrow f_n : X/p^n\mathbb{Z}_p \to X/p^n\mathbb{Z}_p$ is minimal $\forall n \geq 1$.

- **Desjardins and Zieve 1994, Ph.D thesis of Zieve 1996** : induction from level n to level $n+1$.
- **S. Fan and L 2015** : minimal decomposition of power series on finite extensions of $\mathbb{Q}_p$.

Dynamics of

$\phi(x) = ax + 1/x, a \in \mathbb{Q}_p$ with $p \geq 3$

I. $\phi(x) = ax + 1/x, a \in \mathbb{Q}_p$ acting on $\mathbb{P}^1(\mathbb{Q}_p) = \hat{\mathbb{Q}}_p$

Note that

$$\phi(x) - \phi(y) = (a - \frac{1}{xy})(x - y) \quad \text{and} \quad \phi'(x) = a - \frac{1}{x^2}.$$

We distinguish three case : (1) $|a|_p = 1$; (2) $|a|_p > 1$; (3) $|a|_p < 1$.

(1) $|a|_p = 1$: it is easy to see that ϕ has good reduction. The action of ϕ on $\hat{\mathbb{Q}}_p$ is non-expanding. We investigate its minimal decomposition.

(2) $|a|_p > 1$:

- If $\sqrt{-a} \notin \mathbb{Q}_p$, then $\mathcal{J}_\phi = \emptyset$ and $\lim_{n\to\infty} \phi^n(x) = \infty, \forall x \in \mathbb{Q}_p$.
- If $\sqrt{-a} \in \mathbb{Q}_p$, then $\mathcal{J}_\phi \neq \emptyset$.
    - ϕ has two repelling fixed points $x_1 = \frac{1}{\sqrt{1-a}}$ and $x_1 = -\frac{1}{\sqrt{1-a}}$;
    - $(\mathcal{J}_\phi, \phi) \sim (\sum_2, \sigma)$;
    - For $x \in \mathbb{Q}_p \setminus \mathcal{J}_\phi$, $\lim_{n\to\infty} \phi^n(x) = \infty$.

II. $\phi(x) = ax + 1/x, a \in \mathbb{Q}_p$ acting on $\hat{\mathbb{Q}}_p$ – continued

(3) $|a|_p < 1$:
If $v_p(a)$ is odd, then

$$\mathcal{J}_\phi = \{0, \infty\}.$$

If $v_p(a)$ is even, we distinguish two cases :

(a) $\sqrt{-a} \notin \mathbb{Q}_p \Rightarrow \mathcal{J}_\phi = \{0, \infty\}$;

(b) if $\sqrt{-a} \in \mathbb{Q}_p$,

1. if $\sqrt{a} \notin \mathbb{Q}_p$, $\mathcal{J}_\phi$ consists of two parts $\mathcal{J}_1$ and $\mathcal{J}_2$, with $(\mathcal{J}_1, \phi)$ being topologically conjugate to a subshift of finite type and $\mathcal{J}_2$ being the tail of $\mathcal{J}_1$.
2. For the case $\sqrt{a} \notin \mathbb{Q}_p$, there are still lots of work to do.