Some relations between general metrics and ultrametrics

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For any $x \in [0,\, 2)$ consider its decimal expansion $x_0.x_1x_2\dots$ (ten can be replaced by any other radix). If x have two infinite decimal representations ended in zeroes and in nines, respectively, we use first of them. For $x, y \in [0,\, 2)$, $x \neq y$ define

$$\gamma(x, y) = \min\{k \,|\, x_k \neq y_k\},$$

$$d\,(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 10^{-\gamma(x,y)} & \text{otherwise.} \end{cases}$$

It is well known that d_u is an ultrametric. Evidently, $d_r(x, y) \leqslant d(x, y)$ $\forall x, y \in [0,\, 2)$, where d_r is the usual metric, $d_r(x, y) = |x - y|$. Denote $I = [0,\, 1)$,

$$I' = \{x \in I \,|\, x \text{ is not decimal rational}\}.$$

It can be shown that topology generated by d_u and by usual metric are equivalent on X'. The theorem 1 below is some generalisation of this statement.
For a metric space (X, ρ) denote by $\mu_{\rho,H}$ the Hausdorff measure on X generated by the metric ρ.

Theorem

Let (X, d) be a totally bounded metric space. Then there exist ultrametric d_u on X and a set $X' \subset X$ such that
(i) the ultrametric space (X, d_u) is totally bounded;
(ii) $d(x,y) \leq d_u(x,y) \quad \forall x,y \in X$;
(iii) X' is dense in (X, d_u);
(iv) $\mu_{d_u,H}\left(X \diagdown X'\right) = 0$;
(v) The topologies generated by d and d_u on X' coincide. Moreover, there exists a strictly increasing function ζ on $[0,\infty)$ such that

$$d\left(x,y\right) \leqslant \zeta\left(d_u\left(x,y\right)\right) \quad \forall x,y \in X'.$$

Consider the set

$$I^s = \left\{ x = (x_1, \ldots, x_s) \mid x_i \in I,\ i = \overline{1, s} \right\}$$

with the distance

$$d_{r,s}(x, y) = \max_{1 \leqslant i \leqslant s} |x_i - y_i| .$$

Denote $\mathcal{L}^s$ and λ^s the σ-algebra of Lebesgue measurable subsets of I^s and Lebesgue measure on I^s respectively.

Theorem

Let ν be a probability measure on $\mathcal{L}^s$ equivalent to λ. Then there exist an ultrametric d on I^s and a set $A \in \mathcal{L}^s$ such that
(i) the ultrametric space (I^s, d) is totally bounded;
(ii) $d(x, y) \leq d_u(x, y)\ \forall x, y \in X$;
(iii) A is dense in I^s;
(iv) $\nu = \mu_{H,d}$;
(v) $\nu\left(I^n \diagdown A\right) = 0$;
(vi) The topologies generated by d and $d_{r,s}$ on A coincide.

Sometimes when saying about accuracy we say that, for example, “three digits after decimal point are exact” instead of “the error is less than 0.001”. This is justified by the fact that digits are functions of measurement unit and origin that are, in some sense, random. Below we present some mathematical form of this fact.
For $x, y, a \in [0,\, 1)$ set

$$d_{1,a}(x, y) = d(ax, ay).$$

$$d_{2,a}(x, y) = d(a + x, a + y).$$

It can be shown that $d_{1,a}$ and $d_{2,a}$ are ultrametrics for every a and

$$|x - y| = \frac{1}{a}\; |ax - ay| \leqslant \frac{1}{a}\, d(ax, ay) = d_{1,a}(x, y),$$

$$|x - y| \leqslant d_{2,a}(x, y).$$

Denote ξ a random value uniformly distributed on $[0, 1)$.

Theorem

There exist $C_1 > 0$, $C_2 > 0$ such that

$$\mathsf{E}_\xi d(x, y) \leqslant C_k \left| x - y \right|, \; k = 1, 2.$$

This theorem can be naturally extended to the multidimensional case.