Non-Archimedean Operator Algebras

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Spectral theorem. Let $A$ be a bounded linear operator on a Banach space $B$ over a complete algebraically closed non-Archimedean valued field $K$ with a nontrivial valuation; $| \cdot |$ will denote the absolute value in $K$, $O$ is the ring of integers in $K$. We denote by $\| \cdot \|$ both the norm in $B$ and the operator norm $\| A \| = \sup_{x \neq 0} \frac{\| A x \|}{\| x \|}$. We will denote by $\tilde{K}$ the residue field of $K$.

Denote by $\mathcal{L}_A$ the commutative Banach algebra generated by $A$ and the unit operator $I$. $\mathcal{L}_A$ is the closure of the algebra $K[A]$ of polynomials in $A$, with respect to the norm of operators; thus $\mathcal{L}_A$ is a Banach subalgebra of the algebra $L(B)$ of all bounded linear operators. Elements $\lambda \in K$ are identified with the operators $\lambda I$. 
The spectrum $\mathcal{M}(\mathcal{L}_A)$ of the algebra $\mathcal{L}_A$ is defined as the set of all bounded multiplicative seminorms $|\cdot|$ on $\mathcal{L}_A$. $\mathcal{M}(\mathcal{L}_A)$ is provided with the weakest topology, with respect to which all functions $|\cdot| \mapsto |B|, B \in \mathcal{L}_A$, are continuous. In this topology, it is a nonempty Hausdorff compact topological space. If the algebra $\mathcal{L}_A$ is uniform, that is

$$\| T^2 \| = \| T \|^2$$

for any $T \in \mathcal{L}_A$, and all the characters take their values in $K$, then (Berkovich) the space $\mathcal{M}(\mathcal{L}_A)$ is totally disconnected, and $\mathcal{L}_A$ is isomorphic to the algebra $C(\mathcal{M}(\mathcal{L}_A), K)$ of continuous functions on $\mathcal{M}(\mathcal{L}_A)$ with values in $K$. In this case the above isomorphism transforms the characteristic functions $\eta_\Lambda$ of nonempty open-closed subsets $\Lambda \subset \mathcal{M}(\mathcal{L}_A)$ into idempotent operators $E(\Lambda) \in \mathcal{L}_A$, $\| E(\Lambda) \| = 1$. 
These operators form a finitely additive norm-bounded projection-valued measure on the Boolean algebra of open-closed sets, with the non-Archimedean orthogonality property

\[ \| f \| = \sup_{\Lambda} \| E(\Lambda)f \|, \quad f \in B. \]

An operator with the above properties is called *normal*. It is called *strongly normal*, if its spectrum \( \sigma(A) \) is a nonempty totally disconnected compact subset of \( K \), and \( \mathcal{M}(\mathcal{L}_A) = \sigma(A) \).
For a strongly normal operator $A$, we have the spectral decomposition

$$A = \int_{\sigma(A)} \lambda E(d\lambda).$$

For any $\varphi \in C(\sigma(A), K)$ we can define the operator

$$\varphi(A) = \int_{\sigma(A)} \varphi(\lambda) E(d\lambda).$$

The operator $\varphi(A)$ is strongly normal.
The number operator. The model of $p$-adic representation of the canonical commutation relations of quantum mechanics (K., 1996): $\mathcal{B} = C(\mathbb{Z}_p, \mathbb{C}_p)$,

$$(a^+ f)(x) = xf(x - 1), \quad (a^- f)(x) = f(x + 1) - f(x), \quad x \in \mathbb{Z}_p.$$ 

These operators are bounded and satisfy the relation $[a^-, a^+] = I$. Let $A = a^+ a^-$, so that $(Af)(x) = x\{f(x) - f(x - 1)\}$. Then $AP_n = nP_n, n \geq 0$, where

$$P_n(x) = \frac{x(x - 1) \cdots (x - n + 1)}{n!}, \quad n \geq 1; \quad P_0(x) \equiv 1,$$

is the Mahler basis, an orthogonal basis in $C(\mathbb{Z}_p, \mathbb{C}_p) \cong c(\mathbb{Z}_+, \mathbb{C}_p)$. Thus $A$ is normal. An equivalent, though more complicated, construction was given a little earlier by Albeverio and Khrennikov.
Orthoprojections. A projection on a Banach space $B$ is such a linear bounded operator $P$ that $P$ is idempotent: $P^2 = P$. It is obvious that either $P = 0$, or $\|P\| \geq 1$. Kernels of $P \neq 0$ and $I - P$ complement each other having a trivial intersection; if they are orthogonal (in the non-Archimedean sense), then $P$ is called an orthoprojection. In this and only this case, $\|P\| = 1$. For an orthoprojection $P$ different from 0 and $I$, $\|P\| = \|I - P\|$.

**Theorem**

A projection $P$ is an orthoprojection, if and only if it is strongly normal.
**Multiplication operators.** Let us consider the Banach space $\mathcal{B} = C(M, K)$ where $M$ is a compact totally disconnected Hausdorff topological space. It is known that $\mathcal{B}$ possesses an orthonormal basis.

**Theorem**

*Let $A$ be an operator of multiplication on $\mathcal{B}$ by a function $a \in \mathcal{B}$. Then $A$ is strongly normal.*
An operator $U$ on a Banach space $B$ over a complete non-Archimedean valued field $K$ with a nontrivial valuation will be called \textit{unitary}, if $U = I + V$ where $\|V\| < 1$ and $V$ is strongly normal. A unitary operator admits a spectral decomposition

$$U = \int\limits_{\sigma(V)} (1 + \lambda)E_V(d\lambda) = \int\limits_{\sigma(U)} \mu E_U(d\mu).$$

Here $E_V$ is the spectral measure of the operator $V$, the mapping $\varphi(\lambda) = 1 + \lambda$ transforms the spectrum of $V$ into that of $U$, $E_U(M) = E_V(\varphi^{-1}(M))$ for any open-closed subset of $\sigma(U)$. There is an analog of Stone’s theorem about one-parameter groups of unitary operators.
Commutative algebras. Classically, if a $\ast$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space is commutative, then $AA^* = A^*A$ for any $A \in \mathcal{A}$, so that all the operators from $\mathcal{A}$ are normal. Therefore in our situation it is reasonable to consider a commutative algebra $\mathcal{A}$ of normal operators on a Banach space $\mathcal{B}$ over the field $K$. We assume that $\mathcal{A}$ is complete with respect to the norm of operators and contains the unit operator $I$. 
Theorem

Under the above assumptions, the algebra $\mathcal{A}$ is isomorphic to the algebra $C(M, K)$ of $K$-valued continuous functions on a compact totally disconnected Hausdorff topological space $M$. Under this isomorphism, characteristic functions of open-closed subsets $\Lambda \subset M$ correspond to orthoprojections $E(\Lambda)$ forming an orthoprojection-valued finitely additive measure on the algebra of open-closed subsets of $M$. For an operator $F \in \mathcal{A}$ corresponding to a function $f \in C(M, K)$, there is an integral representation

$$F = \int_{M} f(\lambda) E(d\lambda)$$

convergent with respect to the norm of operators.
Algebras with Baer reductions. In order to introduce a possible non-Archimedean counterpart for the class of von Neumann algebras, we need the following reduction procedure. We assume that the space $\mathcal{B}$ is infinite-dimensional and possesses an orthonormal basis (in the non-Archimedean sense). Then $\mathcal{B}$ is isomorphic to the space $c_0(J,K)$ of sequences $x = (x_1, x_2, \ldots, x_i, \ldots)$, $i \in J$, $x_i \in K$, $x_i \to 0$, by the filter of complements to finite subsets of an infinite set $J$. We may assume that $\mathcal{B} = c_0(J,K)$. 
Let $\mathcal{A}$ be an algebra of linear bounded operators on $c_0(J, K)$. Denote by $\mathcal{A}_1$ the closed unit ball in $\mathcal{A}$ – the set of all operators from $\mathcal{A}$ with norm $\leq 1$. $\mathcal{A}_1$ is an algebra over the ring $O$, just as its ideal $\mathcal{A}_0$ consisting of operators of norm $< 1$. The reduced algebra $\tilde{\mathcal{A}} = \mathcal{A}_1/\mathcal{A}_0$ can be considered as a $\tilde{K}$-algebra. Now we can look for a class of $\tilde{K}$-algebras, for which there is a (purely algebraic) theory parallel to the theory of von Neumann algebras. Then the class of algebras $\mathcal{A}$ corresponding to $\tilde{\mathcal{A}}$ from that class will be the desired one.
An algebraic theory of the above kind is the theory of Baer rings and algebras developed by Kaplansky:


A unital ring $R$ is called the Baer ring, if each left (or, equivalently, each right) annihilator in $R$ is generated by an idempotent element. This property was proved by Baer for the ring of all endomorphisms of a vector space of an arbitrary dimension. Kaplansky proved it for any $AW^*$-algebra (the class of $AW^*$-algebras is wider than the class of von Neumann algebras).
A Baer ring $R$ is called Abelian, if all its idempotents are central, and Dedekind finite, if $xy = 1$ implies $yx = 1$. An idempotent $e \in R$ is called Abelian (finite), if the Baer ring $eRe$ is Abelian (respectively, Dedekind finite). If $u$ and $v$ are central idempotents, we write $u \leq v$, if $vu = u$. An idempotent $e$ is called faithful, if the smallest of the central idempotents $v$ satisfying $ve = e$ is equal to $1$.

Kaplansky introduced the following types of Baer rings. A Baer ring $R$ is of type I, if it has a faithful Abelian idempotent. It is of type II, if it has a faithful finite idempotent, but no nonzero Abelian idempotents, and of type III, if it has no nonzero finite idempotents. These classes are subdivided further into finite and infinite ones.
A typical example of a type I Baer ring is the ring of all linear transformations of a vector space of countable dimension. Under some additional conditions, an arbitrary Baer factor of type I is of this form (Wolfson, 1999).

The main result of the theory of Baer rings is *the unique decomposition of every Baer ring into a direct sum of rings of the above types*. 
Let us call an operator algebra $\mathfrak{A}$ an algebra with the Baer reduction, if the reduced algebra $\tilde{\mathfrak{A}}$ is a Baer ring. It is well known that a finite system of elements of norm 1 in a non-Archimedean normed space is orthonormal if and only if their reductions are linearly independent. Therefore the operator ring $\mathfrak{A}_1$ with the Baer reduction is an orthogonal sum of rings with reductions of types I, II, and III.

The simplest example of algebras with the Baer reduction of type I is the algebra of all bounded operators on a Banach space of countable type (that is, $c_0(J, K)$ with a countable set $J$). The very existence of other operator algebras (moreover, factors) with this property is far from obvious. We will present a class of such algebras.
Weak Baer reduction:
Suppose $\tilde{A}$ is not a Baer ring. There is a ring monomorphism $\tilde{A} \rightarrow Q(\tilde{A})$ where $Q(\tilde{A})$ is the maximum right quotient ring. If $Q(\tilde{A})$ is a Baer ring, then its type decomposition also produces an orthogonal decomposition of the ring $A_1$.

The study of various classes of operator algebras with the Baer reduction or the weak Baer reduction seems a huge problem comparable with the whole theory of von Neumann algebras.
Analysis on product spaces. Let $S$ be a totally disconnected compact Hausdorff topological space, $G$ be an Abelian infinite second-countable totally disconnected compact Hausdorff topological group acting transitively on $S$ by homeomorphisms. The action will be denoted as $x \mapsto xa$, $x \in S$, $a \in G$ (the group operation is written multiplicatively). Here we construct and study some operators on $C(S \times G, \mathbb{C}_p)$, the space of continuous functions on $S \times G$ with values in $\mathbb{C}_p$. Here $p$ is a prime number, $\mathbb{C}_p$ is the completion of an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. We will denote $|\cdot|_p$ the absolute value in $\mathbb{C}_p$. Some features of our approach follow the seminal work, F. J. Murray and J. von Neumann, On rings of operators, Ann. Math. 37 (1936), 116–229, though the actual meaning of our objects is different.
We assume that the group $G$ is $p$-compatible. This notion is defined as follows. Denote by $o(G)$ the set of all such natural numbers $n$, for which there exists a subgroup $H \subset G$ with the property that $G/H$ has an element of order $n$. The $p$-compatibility means that $p \notin o(G)$. A typical example is $G = S = \mathbb{Z}_l$ where $l$ is a prime different from $p$.

$p$-Compatible groups possess nice properties resembling those appearing in classical harmonic analysis. $G$ possesses a $\mathbb{C}_p$-valued Haar measure $\mu$. In the Banach space $C(G, \mathbb{C}_p)$, there is an orthonormal basis $\{g_j\}$ consisting of $\mathbb{C}_p$-valued characters, that is maps $G \to \mathbb{C}_p$, such that $g_j(ab) = g_j(a)g_j(b)$. It is convenient to index the characters not by natural numbers but by elements of the dual group $\hat{G}$ consisting of all $\mathbb{C}_p$-valued characters or, equivalently, of all continuous homomorphisms $G \to \mathbb{T}_p$ where $\mathbb{T}_p$ is the set of all roots of 1 in $\mathbb{C}_p$ of orders prime to $p$. 
For groups of this type, $\hat{G}$ is isomorphic also to the Pontryagin dual. The second-countability property of $G$ implies its metrizability and the countability of $\hat{G}$. The group $\hat{G}$ (with the discrete topology) is torsional, that is every finite subset of $G$ lies in a finite subgroup.

The characters are orthonormal not only in the non-Archimedean sense, but also in the integral sense, with respect to the $\mathbb{C}_p$-valued Haar measure $\mu$:

$$\int_G g_j(a)g_n(a^{-1})\mu(da) = \delta_{j,n}, \quad j, n \in \hat{G},$$

where $\delta_{j,n}$ is the Kronecker symbol.
Dealing with a function $F \in C(S \times G, \mathbb{C}_p)$ we can write

$$F(x, a) = \sum_{n \in \hat{G}} \varphi_n(x) g_n(a), \quad x \in S, a \in G,$$

where

$$\varphi_n(x) = \int_G F(x, a) g_n(a^{-1}) \mu(da).$$

The functions $\varphi_n$ are continuous. We use the notation $F \sim \langle \varphi_n(x) \rangle_{n \in \hat{G}}$ where $\|\varphi_n\| \to 0$ by the filter of complements to finite sets in $\hat{G}$ (in such cases we will write $n \to \infty$). This representation implies a matrix representation for operators, the main technical tool of this work.
The crossed product construction

On the Banach space $\mathcal{B} = C(S \times G, \mathbb{C}_p)$, we consider the operators

\[
\overline{U}_{a_0} F(x, a) = F(xa_0, aa_0);
\]
\[
\overline{V}_{a_0} F(x, a) = F(x, a_0^{-1}a);
\]
\[
\overline{W} F(x, a) = F(xa^{-1}, a^{-1});
\]
\[
\overline{L}_\varphi F(x, a) = \varphi(x)F(x, a);
\]
\[
\overline{M}_\varphi F(x, a) = \varphi(xa^{-1})F(x, a),
\]

$x \in S$, $a \in G$. Here $a_0 \in G$ is a fixed element, $\varphi \in C(S, \mathbb{C}_p)$ is a fixed function.

All the above operators are bounded. It is easy to check that

\[
\overline{W} = \overline{W}^{-1}, \quad \overline{W} \overline{U}_{a_0} \overline{W} = \overline{V}_{a_0}, \quad \overline{W} \overline{L}_\varphi \overline{W} = \overline{M}_\varphi.
\]
Denote by $I$ the set of all the operators $\overline{U}_{a_0}$ and $\overline{L}_\varphi$, and by $J$ the set of all the operators $\overline{V}_{a_0}$ and $\overline{M}_\varphi$. The closed linear hull (with respect to the strong operator topology) of the set $I$ will be denoted $R(I)$. Similarly, $R(J)$ is the strongly closed linear hull of $J$. The mapping $A \mapsto \overline{WAW}$ is a spatial isomorphism of the algebras $R(I)$ and $R(J)$. These algebras are the non-Archimedean crossed product algebras, at least for the case of $p$-compatible groups.

**Commutants.**

**Theorem**

(i) $R(J) = I'; R(I) = J'$;

(ii) $R(J) = R(J)''; R(I) = R(I)'''$;

(iii) If the action of $G$ on $S$ is free, then $R(I)$ and $R(J)$ are factors.
Reduction.

Let \( \tilde{\mathcal{A}} \) be the reduced algebra (over the algebraic closure \( \mathcal{F} \) of the finite field \( \mathbb{F}_p \)) corresponding to the algebra \( \mathcal{A} = \mathbb{R}(\mathbf{J}) \).

**Theorem**

\( \tilde{\mathcal{A}} \text{ is a type I Baer ring.} \)
Group algebras

Let $G$ be a discrete group. We consider operators on the space $c_0(G)$ of sequences over a non-Archimedean field $\mathbb{K}$ indexed by elements from $G$ decaying at infinity by the filter of complements to finite sets.

Operators of the right and left regular representations:

\[ U_{a_0}[x_a; \ a \in G] = [x_{aa_0}; \ a \in G]; \]
\[ V_{a_0}[x_a; \ a \in G] = [x_{a_0^{-1}a}; \ a \in G]; \]

$a_0 \in G.$
Let $I$ be the set of all $U_{a_0}$, $J$ be the set of all $V_{a_0}$, $R(I)$ and $R(J)$ their closed hulls with respect to strong (equivalently, uniform) operator topology.

**Theorem**

$R(I) = J'$, $R(J) = I'$.

$I'' = R(J)$, $J'' = R(I)$.

*These closed algebras are factors if and only if $\forall a \in G, a \neq 1$, the set of conjugate elements*

$$L_a = \{c^{-1}ac, c \in G\}$$

*is infinite. The reduction $\tilde{\mathcal{A}}$ for the algebra $\mathcal{A} = R(I)$ is the group algebra $FG$ over the residue field $F$.***
\( \mathcal{A}(G) \) is an analog, simultaneously, of the group von Neumann algebra and the reduced group \( C^* \)-algebra. **An explicit description** of \( \mathcal{A} = \mathcal{A}(G) \):

\( \mathcal{A}(G) \) is isomorphic to the set of sequences \( \eta = [\eta(d) : d \in G] \) tending to zero by the filter of complements to finite sets, with the natural structure of a \( \mathbb{K} \)-Banach space (\( = c_0(G) \)), with the multiplication

\[
(\eta \zeta)(d) = \sum_{l \in G} \eta(l)\zeta(l^{-1}d).
\]

The unit in \( \mathcal{A}(G) \) is the sequence \( e(d) = \begin{cases} 1, & \text{if } d = 1, \\ 0, & \text{otherwise.} \end{cases} \)
Examples (with brief references to the substantiating results from algebra).
1) If $G$ is a finite group, then $\mathcal{A}$ has a type I Baer reduction.
2) If $G$ is a torsion-free polycyclic-by-finite group, then $\mathcal{A}$ has a type I Baer reduction (Cliff, 1980; Faith, 2003).
3) Let $G$ be prime (with no elements of order $p$), locally finite, with countable classes of conjugate elements. Then $\mathcal{A}$ is of weak type III Baer reduction (Hannah and O’Meara, 1977).
4) If $G$ is prime (with no elements of order $p$), solvable, and contains a nontrivial locally finite normal subgroup, then $\mathcal{A}$ is of weak type III Baer reduction (Hannah, 1977, 1979).
5) Let $G$ be a finitary symmetric group, that is a group of permutations of an infinite set $X$ moving only its finite subsets. Then $\mathcal{A}$ is of weak type III Baer reduction (O’Meara, 1980).
$\mathcal{A}(G)$ as a Banach-Hopf algebra

An orthonormal basis in $\mathcal{A}(G)$ (in the non-Archimedean sense):

$$e_x(d) = \begin{cases} 1, & \text{if } d = x, \\ 0, & \text{otherwise.} \end{cases}$$

Comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \hat{\otimes} \mathcal{A}$. For $u = \sum_{x \in G} u(x)e_x$,

$$\Delta(u) = \sum_{x \in G} u(x)(e_x \otimes e_x).$$

The counit:

$$\varepsilon(u) = \sum_{x \in G} u(x), \quad u \in \mathcal{A}(G).$$

The antipode: $S(e_x) = e_{x^{-1}}, \quad x \in G$, with the extension by linearity and continuity.
There is a duality theory – the dual object is a multiplier Banach-Hopf algebra of discrete type.

$\mathcal{A}(G)$ is an example of a non-Archimedean compact quantum group.
Publications

