Non-Archimedean Operator Algebras

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Spectral theorem. Let A be a bounded linear operator on a Banach space $\mathcal{B}$ over a complete algebraically closed non-Archimedean valued field K with a nontrivial valuation; $|\cdot|$ will denote the absolute value in K, O is the ring of integers in K. We denote by $\|\cdot\|$ both the norm in $\mathcal{B}$ and the operator norm $\|A\| = \sup\limits_{x\neq 0} \dfrac{\|Ax\|}{\|x\|}$.

We will denote by $\widetilde{K}$ the residue field of K.
Denote by $\mathcal{L}_A$ the commutative Banach algebra generated by A and the unit operator I. $\mathcal{L}_A$ is the closure of the algebra $K[A]$ of polynomials in A, with respect to the norm of operators; thus $\mathcal{L}_A$ is a Banach subalgebra of the algebra $L(\mathcal{B})$ of all bounded linear operators. Elements $\lambda \in K$ are identified with the operators λI.

The spectrum $\mathcal{M}(\mathcal{L}_A)$ of the algebra $\mathcal{L}_A$ is defined as the set of all bounded multiplicative seminorms $|\cdot|$ on $\mathcal{L}_A$. $\mathcal{M}(\mathcal{L}_A)$ is provided with the weakest topology, with respect to which all functions $|\cdot| \mapsto |B|$, $B \in \mathcal{L}_A$, are continuous. In this topology, it is a nonempty Hausdorff compact topological space. If the algebra $\mathcal{L}_A$ is *uniform*, that is

$$\|T^2\| = \|T\|^2 \text{ for any } T \in \mathcal{L}_A,$$

and all the characters take their values in K, then (Berkovich) the space $\mathcal{M}(\mathcal{L}_A)$ is totally disconnected, and $\mathcal{L}_A$ is isomorphic to the algebra $C(\mathcal{M}(\mathcal{L}_A), K)$ of continuous functions on $\mathcal{M}(\mathcal{L}_A)$ with values in K. In this case the above isomorphism transforms the characteristic functions η_A of nonempty open-closed subsets $\Lambda \subset \mathcal{M}(\mathcal{L}_A)$ into idempotent operators $E(\Lambda) \in \mathcal{L}_A$, $\|E(\Lambda)\| = 1$.

These operators form a finitely additive norm-bounded projection-valued measure on the Boolean algebra of open-closed sets, with the non-Archimedean orthogonality property

$$\|f\| = \sup_{\Lambda} \|E(\Lambda)f\|, \quad f \in \mathcal{B}.$$

An operator with the above properties is called *normal*. It is called *strongly normal*, if its spectrum $\sigma(A)$ is a nonempty totally disconnected compact subset of K, and $\mathcal{M}(\mathcal{L}_A) = \sigma(A)$.

For a strongly normal operator A, we have the spectral decomposition

$$A = \int\limits_{\sigma(A)} \lambda \, E(d\lambda).$$

For any $\varphi \in C(\sigma(A), K)$ we can define the operator

$$\varphi(A) = \int\limits_{\sigma(A)} \varphi(\lambda) \, E(d\lambda).$$

The operator $\varphi(A)$ is strongly normal.

The number operator. The model of p-adic representation of the canonical commutation relations of quantum mechanics (K., 1996): $\mathcal{B} = C(\mathbb{Z}_p, \mathbb{C}_p)$,

$$(a^+ f)(x) = xf(x-1), \quad (a^- f)(x) = f(x+1) - f(x), \quad x \in \mathbb{Z}_p.$$

These operators are bounded and satisfy the relation $[a^-, a^+] = I$. Let $A = a^+ a^-$, so that $(Af)(x) = x\{f(x) - f(x-1)\}$. Then $AP_n = nP_n$, $n \geq 0$, where

$$P_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}, \; n \geq 1; \quad P_0(x) \equiv 1,$$

is the Mahler basis, an orthogonal basis in $C(\mathbb{Z}_p, \mathbb{C}_p) \cong c(\mathbb{Z}_+, \mathbb{C}_p)$. Thus A is normal. An equivalent, though more complicated, construction was given a little earlier by Albeverio and Khrennikov.

Orthoprojections. A projection on a Banach space $\mathcal{B}$ is such a linear bounded operator P that P is idempotent: $P^2 = P$. It is obvious that either $P = 0$, or $\|P\| \geq 1$. Kernels of $P \neq 0$ and $I - P$ complement each other having a trivial intersection; if they are orthogonal (in the non-Archimedean sense), then P is called an *orthoprojection*. In this and only this case, $\|P\| = 1$. For an orthoprojection P different from 0 and I, $\|P\| = \|I - P\|$.

Theorem

A projection P is an orthoprojection, if and only if it is strongly normal.

Multiplication operators. Let us consider the Banach space $\mathcal{B} = C(M, K)$ where M is a compact totally disconnected Hausdorff topological space. It is known that $\mathcal{B}$ possesses an orthonormal basis.

Theorem

Let A be an operator of multiplication on $\mathcal{B}$ by a function $a \in \mathcal{B}$. Then A is strongly normal.

An operator U on a Banach space $\mathcal{B}$ over a complete non-Archimedean valued field K with a nontrivial valuation will be called *unitary*, if $U = I + V$ where $\|V\| < 1$ and V is strongly normal. A unitary operator admits a spectral decomposition

$$U = \int\limits_{\sigma(V)} (1+\lambda) E_V(d\lambda) = \int\limits_{\sigma(U)} \mu E_U(d\mu).$$

Here E_V is the spectral measure of the operator V, the mapping $\varphi(\lambda) = 1 + \lambda$ transforms the spectrum of V into that of U, $E_U(M) = E_V(\varphi^{-1}(M))$ for any open-closed subset of $\sigma(U)$. There is an analog of Stone's theorem about one-parameter groups of unitary operators.

Commutative algebras. Classically, if a $*$-algebra $\mathfrak{A}$ of bounded operators on a Hilbert space is commutative, then $AA^* = A^*A$ for any $A \in \mathfrak{A}$, so that all the operators from $\mathfrak{A}$ are normal. Therefore in our situation it is reasonable to consider a commutative algebra $\mathfrak{A}$ of normal operators on a Banach space $\mathcal{B}$ over the field K. We assume that $\mathfrak{A}$ is complete with respect to the norm of operators and contains the unit operator I.

Theorem

Under the above assumptions, the algebra $\mathfrak{A}$ is isomorphic to the algebra $C(M,K)$ of K-valued continuous functions on a compact totally disconnected Hausdorff topological space M. Under this isomorphism, characteristic functions of open-closed subsets $\Lambda \subset M$ correspond to orthoprojections $E(\Lambda)$ forming an orthoprojection-valued finitely additive measure on the algebra of open-closed subsets of M. For an operator $F \in \mathfrak{A}$ corresponding to a function $f \in C(M,K)$, there is an integral representation

$$F = \int\limits_M f(\lambda)\, E(d\lambda)$$

convergent with respect to the norm of operators.

Algebras with Baer reductions. In order to introduce a possible non-Archimedean counterpart for the class of von Neumann algebras, we need the following reduction procedure. We assume that the space $\mathcal{B}$ is infinite-dimensional and possesses an orthonormal basis (in the non-Archimedean sense). Then $\mathcal{B}$ is isomorphic to the space $c_0(J,K)$ of sequences $x=(x_1,x_2,\ldots,x_i,\ldots)$, $i\in J$, $x_i\in K$, $x_i\to 0$, by the filter of complements to finite subsets of an infinite set J. We may assume that $\mathcal{B}=c_0(J,K)$.

Let $\mathfrak{A}$ be an algebra of linear bounded operators on $c_0(J, K)$. Denote by $\mathfrak{A}_1$ the closed unit ball in $\mathfrak{A}$ – the set of all operators from $\mathfrak{A}$ with norm ≤ 1. $\mathfrak{A}_1$ is an algebra over the ring O, just as its ideal $\mathfrak{A}_0$ consisting of operators of norm < 1. The *reduced algebra* $\widetilde{\mathfrak{A}} = \mathfrak{A}_1/\mathfrak{A}_0$ can be considered as a $\widetilde{K}$-algebra. Now we can look for a class of $\widetilde{K}$-algebras, for which there is a (purely algebraic) theory parallel to the theory of von Neumann algebras. Then the class of algebras $\mathfrak{A}$ corresponding to $\widetilde{\mathfrak{A}}$ from that class will be the desired one.

An algebraic theory of the above kind is the theory of Baer rings and algebras developed by Kaplansky:

I. Kaplansky, *Rings of Operators*, W. A. Benjamin, New York, 1968.

A unital ring R is called the Baer ring, if each left (or, equivalently, each right) annihilator in R is generated by an idempotent element. This property was proved by Baer for the ring of all endomorphisms of a vector space of an arbitrary dimension. Kaplansky proved it for any AW^*-algebra (the class of AW^*-algebras is wider than the class of von Neumann algebras).

A Baer ring R is called Abelian, if all its idempotents are central, and Dedekind finite, if $xy = 1$ implies $yx = 1$. An idempotent $e \in R$ is called Abelian (finite), if the Baer ring eRe is Abelian (respectively, Dedekind finite). If u and v are central idempotents, we write $u \leq v$, if $vu = u$. An idempotent e is called faithful, if the smallest of the central idempotents v satisfying $ve = e$ is equal to 1.

Kaplansky introduced the following types of Baer rings. A Baer ring R is of type I, if it has a faithful Abelian idempotent. It is of type II, if it has a faithful finite idempotent, but no nonzero Abelian idempotents, and of type III, if it has no nonzero finite idempotents. These classes are subdivided further into finite and infinite ones.

A typical example of a type I Baer ring is the ring of all linear transformations of a vector space of countable dimension. Under some additional conditions, an arbitrary Baer factor of type I is of this form (Wolfson, 1999).

The main result of the theory of Baer rings is *the unique decomposition of every Baer ring into a direct sum of rings of the above types*.

Let us call an operator algebra $\mathfrak{A}$ *an algebra with the Baer reduction*, if the reduced algebra $\widetilde{\mathfrak{A}}$ is a Baer ring. It is well known that a finite system of elements of norm 1 in a non-Archimedean normed space is orthonormal if and only if their reductions are linearly independent. Therefore **the operator ring $\mathfrak{A}_1$ with the Baer reduction is an orthogonal sum of rings with reductions of types I, II, and III**.

The simplest example of algebras with the Baer reduction of type I is the algebra of all bounded operators on a Banach space of countable type (that is, $c_0(J, K)$ with a countable set J). The very existence of other operator algebras (moreover, factors) with this property is far from obvious. We will present a class of such algebras.

Weak Baer reduction:
Suppose $\widetilde{\mathfrak{A}}$ is not a Baer ring. There is a ring monomorphism $\widetilde{\mathfrak{A}} \to Q(\widetilde{\mathfrak{A}})$ where $Q(\widetilde{\mathfrak{A}})$ is the maximum right quotient ring. If $Q(\widetilde{\mathfrak{A}})$ is a Baer ring, then its type decomposition also produces an orthogonal decomposition of the ring $\mathfrak{A}_1$.
The study of various classes of operator algebras with the Baer reduction or the weak Baer reduction seems a huge problem comparable with the whole theory of von Neumann algebras.

Analysis on product spaces. Let S be a totally disconnected compact Hausdorff topological space, G be an Abelian infinite second-countable totally disconnected compact Hausdorff topological group acting transitively on S by homeomorphisms. The action will be denoted as $x \mapsto xa$, $x \in S$, $a \in G$ (the group operation is written multiplicatively). Here we construct and study some operators on $C(S \times G, \mathbb{C}_p)$, the space of continuous functions on $S \times G$ with values in $\mathbb{C}_p$. Here p is a prime number, $\mathbb{C}_p$ is the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. We will denote $|\cdot|_p$ the absolute value in $\mathbb{C}_p$. Some features of our approach follow the seminal work,

F. J. Murray and J. von Neumann, On rings of operators, *Ann. Math.* **37** (1936), 116–229,

though the actual meaning of our objects is different.

We assume that **the group G is p-compatible**. This notion is defined as follows. Denote by $o(G)$ the set of all such natural numbers n, for which there exists a subgroup $H \subset G$ with the property that G/H has an element of order n. The p-compatibility means that $p \notin o(G)$. A typical example is $G = S = \mathbb{Z}_l$ where l is a prime different from p.

p-Compatible groups possess nice properties resembling those appearing in classical harmonic analysis. G possesses a $\mathbb{C}_p$-valued Haar measure μ. In the Banach space $C(G, \mathbb{C}_p)$, there is an orthonormal basis $\{g_j\}$ consisting of $\mathbb{C}_p$-valued characters, that is maps $G \to \mathbb{C}_p$, such that $g_j(ab) = g_j(a)g_j(b)$. It is convenient to index the characters not by natural numbers but by elements of the dual group $\widehat{G}$ consisting of all $\mathbb{C}_p$-valued characters or, equivalently, of all continuous homomorphisms $G \to \mathbb{T}_p$ where $\mathbb{T}_p$ is the set of all roots of 1 in $\mathbb{C}_p$ of orders prime to p.

For groups of this type, $\widehat{G}$ is isomorphic also to the Pontryagin dual. The second-countability property of G implies its metrizability and the countability of $\widehat{G}$. The group $\widehat{G}$ (with the discrete topology) is torsional, that is every finite subset of G lies in a finite subgroup.

The characters are orthonormal not only in the non-Archimedean sense, but also in the integral sense, with respect to the $\mathbb{C}_p$-valued Haar measure μ:

$$\int\limits_G g_j(a) g_n(a^{-1})\, \mu(da) = \delta_{j,n}, \quad j, n \in \widehat{G},$$

where $\delta_{j,n}$ is the Kronecker symbol.

Dealing with a function $F \in C(S \times G, \mathbb{C}_p)$ we can write

$$F(x,a) = \sum_{n \in \widehat{G}} \varphi_n(x) g_n(a), \quad x \in S, a \in G,$$

where

$$\varphi_n(x) = \int_G F(x,a) g_n(a^{-1}) \, \mu(da).$$

The functions φ_n are continuous. We use the notation $F \sim \langle \varphi_n(x) \rangle_{n \in \widehat{G}}$ where $\|\varphi_n\| \to 0$ by the filter of complements to finite sets in $\widehat{G}$ (in such cases we will write $n \to \infty$). This representation implies a matrix representation for operators, the main technical tool of this work.

The crossed product construction

. On the Banach space $\mathcal{B} = C(S \times G, \mathbb{C}_p)$, we consider the operators

$$\overline{U}_{a_0} F(x,a) = F(xa_0, aa_0);$$
$$\overline{V}_{a_0} F(x,a) = F(x, a_0^{-1} a);$$
$$\overline{W} F(x,a) = F(xa^{-1}, a^{-1});$$
$$\overline{L}_{\varphi} F(x,a) = \varphi(x) F(x,a);$$
$$\overline{M}_{\varphi} F(x,a) = \varphi(xa^{-1}) F(x,a),$$

$x \in S$, $a \in G$. Here $a_0 \in G$ is a fixed element, $\varphi \in C(S, \mathbb{C}_p)$ is a fixed function.

All the above operators are bounded. It is easy to check that

$$\overline{W} = \overline{W}^{-1},\ \overline{W}\,\overline{U}_{a_0}\overline{W} = \overline{V}_{a_0},\ \overline{W}\,\overline{L}_{\varphi}\overline{W} = \overline{M}_{\varphi}.$$

Denote by $\mathbf{I}$ the set of all the operators $\overline{U}_{a_0}$ and $\overline{L}_\varphi$, and by $\mathbf{J}$ the set of all the operators $\overline{V}_{a_0}$ and $\overline{M}_\varphi$. The closed linear hull (with respect to the strong operator topology) of the set $\mathbf{I}$ will be denoted $\mathbf{R}(\mathbf{I})$. Similarly, $\mathbf{R}(\mathbf{J})$ is the strongly closed linear hull of $\mathbf{J}$. The mapping $\overline{A} \mapsto \overline{W}\,\overline{A}\,\overline{W}$ is a spatial isomorphism of the algebras $\mathbf{R}(\mathbf{I})$ and $\mathbf{R}(\mathbf{J})$. These algebras are the non-Archimedean crossed product algebras, at least for the case of p-compatible groups.

Commutants.

Theorem

(i) $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$; $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$;

(ii) $\mathbf{R}(\mathbf{J}) = \mathbf{R}(\mathbf{J})''$; $\mathbf{R}(\mathbf{I}) = \mathbf{R}(\mathbf{I})''$;

(iii) *If the action of G on S is free, then* $\mathbf{R}(\mathbf{I})$ *and* $\mathbf{R}(\mathbf{J})$ *are factors.*

Reduction.

Let $\widetilde{\mathfrak{A}}$ be the reduced algebra (over the algebraic closure $\mathcal{F}$ of the finite field $\mathbb{F}_p$) corresponding to the algebra $\mathfrak{A} = \mathbf{R}(\mathbf{J})$.

Theorem

$\widetilde{\mathfrak{A}}$ *is a type I Baer ring.*

Group algebras

Let G be a discrete group. We consider operators on the space $c_0(G)$ of sequences over a non-Archimedean field $\mathbb{K}$ indexed by elements from G decaying at infinity by the filter of complements to finite sets.

Operators of the right and left regular representations:

$$U_{a_0}[x_a;\ a \in G] = [x_{aa_0};\ a \in G];$$

$$V_{a_0}[x_a;\ a \in G] = [x_{a_0^{-1}a};\ a \in G];$$

$a_0 \in G$.

Let $\mathbf{I}$ be the set of all U_{a_0}, $\mathbf{J}$ be the set of all V_{a_0}, $\mathbf{R}(\mathbf{I})$ and $\mathbf{R}(\mathbf{J})$ their closed hulls with respect to strong (equivalently, uniform) operator topology.

Theorem

$\mathbf{R}(\mathbf{I}) = \mathbf{J}'$, $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$.
$\mathbf{I}'' = \mathbf{R}(\mathbf{J})$, $\mathbf{J}'' = \mathbf{R}(\mathbf{I})$.
These closed algebras are factors if and only if $\forall a \in G, a \neq 1$, *the set of conjugate elements*

$$\mathcal{L}_a = \{c^{-1}ac, c \in G\}$$

is infinite. The reduction $\widetilde{\mathfrak{A}}$ *for the algebra* $\mathfrak{A} = R(\mathbf{I})$ *is the group algebra* $\mathcal{F}G$ *over the residue field* $\mathcal{F}$.

$\mathfrak{A}(G)$ is an analog, simultaneously, of the group von Neumann algebra and the reduced group C^*-algebra. **An explicit description** of $\mathfrak{A} = \mathfrak{A}(G)$:

$\mathfrak{A}(G)$ is isomorphic to the set of sequences $\eta = [\eta(d) : \ d \in G]$ tending to zero by the filter of complements to finite sets, with the natural structure of a $\mathbb{K}$-Banach space ($= c_0(G)$), with the multiplication

$$(\eta\zeta)(d) = \sum_{l \in G} \eta(l)\zeta(l^{-1}d).$$

The unit in $\mathfrak{A}(G)$ is the sequence $e(d) = \begin{cases} 1, & \text{if } d = 1, \\ 0, & \text{otherwise.} \end{cases}$

Examples (with brief references to the substantiating results from algebra).

1) If G is a finite group, then $\mathfrak{A}$ has a type I Baer reduction.

2) If G is a torsion-free polycyclic-by-finite group, then $\mathfrak{A}$ has a type I Baer reduction (Cliff, 1980; Faith, 2003).

3) Let G be prime (with no elements of order p), locally finite, with countable classes of conjugate elements. Then $\mathfrak{A}$ is of weak type III Baer reduction (Hannah and O'Meara, 1977).

4) If G is prime (with no elements of order p), solvable, and contains a nontrivial locally finite normal subgroup, then $\mathfrak{A}$ is of weak type III Baer reduction (Hannah, 1977, 1979).

5) Let G be a finitary symmetric group, that is a group of permutations of an infinite set X moving only its finite subsets. Then $\mathfrak{A}$ is of weak type III Baer reduction (O'Meara, 1980).

$\mathfrak{A}(G)$ as a Banach-Hopf algebra

An orthonormal basis in $\mathfrak{A}(G)$ (in the non-Archimedean sense):

$$e_x(d) = \begin{cases} 1, & \text{if } d = x, \\ 0, & \text{otherwise.} \end{cases}$$

Comultiplication $\Delta : \mathfrak{A} \to \mathfrak{A}\hat{\otimes}\mathfrak{A}$. For $u = \sum\limits_{x\in G} u(x)e_x$,

$$\Delta(u) = \sum_{x\in G} u(x)(e_x \otimes e_x).$$

The counit:

$$\varepsilon(u) = \sum_{x\in G} u(x), \quad u \in \mathfrak{A}(G).$$

The antipode: $S(e_x) = e_{x^{-1}}, \quad x \in G$, with the extension by linearity and continuity.

There is a duality theory – the dual object is a multiplier Banach-Hopf algebra of discrete type.

$\mathfrak{A}(G)$ is an example of a **non-Archimedean compact quantum group**.

Publications

1. A. N. Kochubei, Non-Archimedean unitary operators, *Methods of Functional Analysis and Topology*, 17, No. 3 (2011), 219–224.
2. A. N. Kochubei, On some classes of non-Archimedean operator algebras, *Contemporary Math.* 596 (2013), 133–148.
3. A. N. Kochubei, Non-Archimedean group algebras with Baer reductions, *Algebras and Represent. Theory* 17, No. 6 (2014), 1861–1867.