A p-adic probability logic

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Outline

1. Probability logic

2. $L_{Q_p}$

3. $CPL_{Q_p}^{fin}$, $CPL_{Z_p}$

4. $L_{Q_p}^D$
Basic concepts

• Weighted logics
• Predicate and operator logics
• Completeness
• Decidability and complexity
Issues related to range

- Noncompactness: $\{Pr(\alpha) \neq r : r \in X\}$ is finitely satisfiable for any infinite $X$;
- Completeness fails for uncountable languages.
**Definition**

Let $r = p^n$ ($n \in \mathbb{Z}$), $K[0, r] = \{x \in \mathbb{Q}_p : \|x\|_p \leq r\}$.

A $p$–adic $r$–probability space is any structure $(W, H, \mu)$ such that $W \neq \emptyset$, $H \leq \mathcal{P}(W)$ and $\mu : H \rightarrow K[0, r]$ is additive.

- **Boundedness condition:** for every $A \in H$

  $$\|A\| = \sup \{|\mu(B)|_p : B \in A, B \subset A\} < \infty$$
Basic $p$-adic probability logic $L_{Qp}$

Enriches propositional calculus with the operators:

$$K_{r,\rho}\alpha$$

with the intended meaning:

“the probability of $\alpha$ belongs to the $p$-adic ball with the center $r$ and the radius $\rho$”
Formal language

Let $p$ be a fixed prime and $M \in \mathbb{N}$ be an arbitrary large but fixed positive integer. We introduce the following sets:

1. $\mathbb{Q}_M = \{ r \in \mathbb{Q} : |r|_p \leq p^M \}$;
2. $\mathbb{Z}_M = \mathbb{Z}^- \cup \{0, 1, 2, \ldots, M\}$, where $\mathbb{Z}^-$ denotes the set of all negative integers;
3. $R_M = \{ p^{M-n} : n \in \mathbb{N} \} \cup \{0\} = \{ p^n : n \in \mathbb{Z}_M \} \cup \{0\}$. 
Formal language

- \( \text{Var} = \{p_1, p_n, \ldots \} \): countable set of propositional letters, connectives \( \neg \) and \( \land \);
- Probability operators \( K_{r,\rho}, r \in \mathbb{Q}_M, \rho \in \mathbb{R}_M \);
- \( \text{For}_{CI} \)-the set of classical propositional formulas over \( \text{Var} \);

Definition

The set \( \text{For}_P \) of all probabilistic formulas is defined as the least set satisfying the following conditions:

- If \( \alpha, \beta \in \text{For}_{CI}, r \in \mathbb{Q}_M, \rho \in \mathbb{R} \) then \( K_{r,\rho}\alpha, \beta \) is probabilistic formula.
- If \( \varphi, \phi \) are probabilistic formulas, then \( \neg \varphi \), \( \varphi \land \phi \) are probabilistic formulas.
Semantics

Definition
An $L_{Q_p}$-model is a structure $\mathcal{M} = \langle W, H, \mu, \nu \rangle$ where:

- $W$ is a nonempty set of elements called worlds;
- $H$ is an algebra of subsets of $W$;
- $\mu : H \rightarrow K[0, p^M]$ is a measure (additive function) such that $\mu(W) = 1$;
- $\nu : W \times \text{Var} \rightarrow \{true, false\}$ is a valuation which associated with every world $w \in W$ a truth assignment $\nu(w, \cdot)$ on propositional letters; the valuation $\nu(w, \cdot)$ is extended to classical propositional formulas as usual.

If $\mathcal{M}$ is an $L_{Q_p}$-model, by $[\alpha]_\mathcal{M}$ we denote the set $w$ such that $\nu(w, \alpha) = \text{true}$. 
Satisfiability

Definition

Let $\mathcal{M} = \langle W, H, \mu, v \rangle$ be an $L_{Q_p}$-model. The satisfiability relation is inductively defined as follows:

- If $\alpha \in \text{For}_{Cl}$ then $\mathcal{M} \models \alpha$ iff $v(w, \alpha) = \text{true}$ for all $w \in W$;
- If $\alpha \in \text{For}_{Cl}$ then $\mathcal{M} \models K_{r,\rho}\alpha$ iff $|\mu([\alpha]) - r|_p \leq \rho$;
- If $\varphi \in \text{For}_P$ then $\mathcal{M} \models \neg \varphi$ iff it is not $\mathcal{M} \models \varphi$;
- If $\varphi, \psi \in \text{For}_P$ then $\mathcal{M} \models \varphi \land \psi$ iff $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$. 
Axiomatization

The axiom system $AX_{L_{Q_p}}$ of the logic $L_{Q_p}$ contains the following axioms and inference rules:

**Axioms**

A1 Substitutional instances of tautologies;

A2 $K_{r_1, \rho_1} \alpha \Rightarrow K_{r_2, \rho_2} \alpha$, whenever $\rho' \geq \rho$;

A3 $K_{r_1, \rho_1} \alpha \land K_{r_2, \rho_2} \beta \land K_{0,0}(\alpha \land \beta) \Rightarrow K_{r_1 + r_2, \max(\rho_1, \rho_2)}(\alpha \lor \beta)$;

A4 $K_{r_1, \rho_1} \alpha \Rightarrow \neg K_{r_2, \rho_2} \alpha$, if $|r_1 - r_2|_p > \max(\rho_1, \rho_2)$;

A5 $K_{r_1, \rho_1} \alpha \Rightarrow K_{r_2, \rho_2} \alpha$, if $|r_1 - r_2|_p \leq \rho$;
Inference rules

R1  Modus ponens: From $\Phi$ and $\Phi \Rightarrow \Psi$ infer $\Psi$;
R2  Necessitation (a): from $\alpha$ infer $K_{1,0}\alpha$;
R3  Necessitation (b): from $\neg\alpha$ infer $K_{0,0}\alpha$;
R4  Coherence: From $\alpha \iff \beta$ infer $K_{r,\rho}\alpha \Rightarrow K_{r,\rho}\beta$;
R5  Range: From $\phi \Rightarrow \neg K_{r,\rho}^{M-n}\alpha$ for all $r \in \mathbb{Q}_M$, infer $\neg\phi$;
R6  Convergence: From $\phi \Rightarrow K r, \rho\alpha$ for all $\rho \in R_M$, infer $\phi \Rightarrow K_{r,0}\alpha$. 
Soundness

Theorem
The axiomatic system $AX_{L_{Q_p}}$ is sound with respect to the class of $L_{Q_p}$-models.

Theorem
(Deduction theorem) Let $T$ be a set of formulas and $A$ and $B$ both classical or both propositional formulas. Then, $T, A \vdash B$ implies $T \vdash A \Rightarrow B$. 
Completeness

Theorem

*Every consistent set of formulas $T$ can be extended to a maximal consistent set.*
Completeness

- $T^*$ maximal consistent set of formulas;
- $T^*$ satisfies: for every formula $\alpha \in \text{For}_\text{Cl}$ and every $m \in \mathbb{N}$ there are countably many $r \in \mathbb{Q}_M$ such that $K_{r, p^{M-m}\alpha} \in T^*$;
- For every $\alpha \in \text{For}_\text{Cl}$ we define sequence $r_m$ in the following way:
  - For every $m \in \mathbb{N}$ we arbitrarily choose $r$ such that $K_{r, p^{M-m}\alpha} \in T^*$ and this $r$ will be $m$-th number of the sequence, i.e., $r_m = r$.
  - For $\alpha \in \text{For}_\text{Cl}$ we obtain sequence $r(\alpha) = r_0, r_1, \ldots$, where $K_{r_j, p^{M-j}\alpha} \in T^*$.

Lemma

Let $r(\alpha)$ be defined as above. Then $r(\alpha)$ is a Cauchy sequence with respect to the $p$-adic norm. Moreover, the limit of $r(\alpha)$ does not depend on the choice of $r_k$'s.
Completeness-canonical model

Let $M_{T^*} = \langle W, H, \mu, \nu \rangle$, where:

- $T$ is consistent set, $\bar{T}$ is set of all classical consequences of $T$
- $W = \{ w | \nu(w, \alpha) = true \text{ for every } \alpha \in \bar{T} \}$ contains all classical propositional interpretations that satisfy $\bar{T}$,
- $H = \{ [\alpha] : \alpha \in For_{Cl} \}$
- $\mu : H \rightarrow \mathbb{Q}_p$: Let $r(\alpha) = (r_n)_{n \in \mathbb{N}}$. Then

$$\mu([\alpha]) = \begin{cases} r & \text{if } K_{r,0}\alpha \in T^* \\ \lim_{n \rightarrow \infty} r_n & \text{otherwise} \end{cases}$$

- for every world $w$ and every $p \in Var$, $\nu(w, p) = true$ iff $w \models p$. 
Completeness, Decidability

**Theorem**
A set of formulas $T$ is consistent iff has an $L_{Q_p}$-model.

**Theorem**
The satisfiability problem for $L_{Q_p}$ formulas is decidable.
Satisfying boundedness condition

- **boundedness condition:** If $F$ is a field of subsets of some set $\Omega$ then, for every $A \in F$

\[ \sup\{|\mu(B)|_p : B \in F, B \subset A\} < \infty \]

- This condition can be ensured by reducing the range of probabilities to an arbitrarily large (but fixed) ball $K[0, p^M]$, where $M$ is some fixed integer.

- $P(A|B) \cdot P(B) = P(A \land B)$; $K[0, p^M]$ is not closed for multiplication!
Satisfying boundedness condition

- We might proceed in two ways.
  1. Using unit ball $K[0,1]$ as a range of probability, since it closed for multiplication-$CPL_{\mathbb{Z}_p}$
  2. To built formulas from the finite set of propositional letters, but to retain $\mathbb{Q}_p$ as a range of probability-$CPL_{\mathbb{Q}_p}^{fin}$.

In this way we compute supremum of finitely many numbers of the form $p^n$, $n \in \mathbb{Z}$, which is again a finite number, precisely:

- **boundedness condition**: For every $\alpha \in For_{Cl}$ in the model $M = \langle W, H, \mu, \nu \rangle$

\[
\sup \{|\mu([\beta])|_p : [\beta] \in H, [\beta] \subset [\alpha]\} < \infty
\]

- For every $\alpha \in For_{Cl}$ there exist finitely many logically inequivalent formulas $\beta$ such that $\beta \Rightarrow \alpha$ is tautology, i.e., such that $[\beta] \subset [\alpha]$. 
Formal language

\( CK_{r,\rho} \alpha, \beta \): “the conditional probability of \( \alpha \) given \( \beta \) is in the p-adic ball with the center \( r \) and the radius \( \rho \)”

- \( CPL_{\mathbb{Q}_p}^{\text{fin}} \): \( \text{Var} = \{p_1, \ldots, p_n\} \);
- \( CPL_{\mathbb{Z}_p} \)
  - \( \mathbb{Q}_1 = \{ r \in \mathbb{Q} \mid |r|_p \leq 1 \} \),
  - \( R = \{ p^{-n} \mid n \in \mathbb{N} \} \cup \{0\} \),
  - \( (CK_{r,\rho})_{r \in \mathbb{Q}_1, \rho \in R} \).
- \( CPL_{\mathbb{Q}_p}^{\text{fin}} \)
  - \( R_1 = \{ p^n \mid n \in \mathbb{Z} \} \cup \{0\} \)
  - \( CK_{r,\rho} \ r \in \mathbb{Q}, \rho \in R_1 \)
Semantics

Definition

A $\text{CPL}_{\mathbb{Z}_p},(\text{CPL}^\text{fin}_{\mathbb{Q}_p})$-model is a structure $M = \langle W, H, \mu, v \rangle$ where:

- $W$ is a nonempty set of elements called worlds.
- $H$ is an algebra of subsets of $W$.
- $\mu : H \to \mathbb{Z}_p(\mu : H \to \mathbb{Q}_p)$ is a measure (additive function) such that $\mu(W) = 1$.
- $v : W \times \text{Var} \to \{\text{true}, \text{false}\}$ is a valuation which associates with every world $w \in W$ a truth assignment $v(w, \cdot)$ on propositional letters; the valuation $v(w, \cdot)$ is extended to classical propositional formulas as usual.
Satisfiability

Definition
Let $M = \langle W, H, \mu, \nu \rangle$ be an $CPL_{\mathbb{Z}_p}$, $(CPL_{\mathbb{Q}_p}^{\text{fin}})$-model. The satisfiability relation is inductively defined as follows:

- If $\alpha \in For_{Cl}$ then $M \models \alpha$ iff $\nu(w, \alpha) = \text{true}$ for every $w \in W$.
- If $\alpha, \beta \in For_{Cl}$ then $M \models CK_{r, \rho} \alpha, \beta$ iff:
  - $\mu([\beta]) = 0$ and $|r - 1|_p \leq \rho$ or
  - $\mu([\beta]) \neq 0$ and $|\frac{\mu([\alpha \land \beta])}{\mu([\beta])} - r|_p \leq \rho$.
- If $\varphi \in For_P$, then $M \models \neg \varphi$ iff it is not $M \models \varphi$.
- If $\varphi, \psi \in For_P$ then $M \models \varphi \land \psi$ iff $M \models \varphi$ and $M \models \psi$. 
Satisfiability

- \( M \models CK_{r,\rho} \alpha, \top \) iff \( |\mu([\alpha]) - r|_p \leq \rho \).
- Conditional probability, \( P(\alpha|T) \) comes to standard probability \( P(\alpha) \).
- \( CK_{r,\rho} \alpha, \top \) will be denoted by \( K_{r,\rho} \alpha \).
Axiomatization

1. Substitutional instances of tautologies.

2. \( K_{r_1,\rho_1} \alpha \land K_{r_2,\rho_2} \beta \land K_{0,0}(\alpha \land \beta) \Rightarrow K_{r_1 + r_2, \max(\rho_1, \rho_2)}(\alpha \lor \beta) \).

3. \( CK_{r,\rho} \alpha, \beta \Rightarrow CK_{r,\rho'} \alpha, \beta, \; \rho' \geq \rho \)

4. \( CK_{r_1,\rho_1} \alpha, \beta \Rightarrow \neg CK_{r_2,\rho_2} \alpha, \beta, \; \text{if} \; |r_1 - r_2|_p > \max(\rho_1, \rho_2) \),

5. \( CK_{r_1,\rho} \alpha, \beta \Rightarrow CK_{r_2,\rho} \alpha, \beta, \; \text{if} \; |r_1 - r_2|_p \leq \rho \)

6. \( CPL_{\mathbb{Z}_p} \) \( K_{r_1 r_2,\rho_1}(\alpha \land \beta) \land K_{r_2,\rho_2} \beta \Rightarrow CK_{r_1, \frac{\max\{\rho_1, \rho_2\}}{|r_2|_p}} \alpha, \beta \) whenever

\[ r_2 \neq 0, \; |r_1|_p \leq 1, \; |r_2|_p > \rho_2; \]

7. \( CPL_{\mathbb{Q}_p}^{fin} \) \( K_{r_1 r_2,\rho_1}(\alpha \land \beta) \land K_{r_2,\rho_2} \beta \Rightarrow CK_{r_1, \frac{\max\{\rho_1, |r_1|_p \cdot \rho_2\}}{|r_2|_p}} \alpha, \beta \)

\[ r_2 \neq 0, \; |r_2|_p > \rho_2 \]

8. \( CK_{r,\rho} \alpha, \beta \land K_{r_1,\rho_1} \beta \Rightarrow K_{r \cdot r_1, \max\{|r_1|_p \cdot \rho, |r|_p \cdot \rho_1\}} \alpha \land \beta, \; \text{if} \)

\[ r_1 \neq 0, \; |r_1|_p > \rho_1. \]

9. \( K_{0,0} \beta \land K_{r,\rho}(\alpha \land \beta) \Rightarrow CK_{1,0} \alpha, \beta \)
Inference rules

1. From \( A \) and \( A \Rightarrow B \) infer \( B \). Here \( A \) and \( B \) are both propositional, or both probabilistic formulas.

2. From \( \alpha \) infer \( K_{1,0}\alpha \)

3. If \( n \in \mathbb{N} \), from \( \varphi \Rightarrow \neg K_{r,p^{-n}}\alpha \) for every \( r \in \mathbb{Q} \), infer \( \varphi \Rightarrow \bot \).

4. From \( \alpha \Rightarrow \bot \), infer \( K_{0,0}\alpha \)

5. If \( r \in \mathbb{Q} \), from \( \varphi \Rightarrow CK_{r,p^{-n}}\alpha, \beta \) for every \( n \in \mathbb{N} \), infer \( \varphi \Rightarrow CK_{r,0}\alpha, \beta \).

6. From \( \alpha \Leftrightarrow \beta \) infer \( (K_{r,\rho}\alpha \Leftrightarrow K_{r,\rho}\beta) \).
**Formal language, Semantics**

\[ D_\rho \alpha, \beta : \text{"the p-adic distance between the probabilities of } \alpha \text{ and } \beta \text{ is less than or equal to } \rho \" } \]

- \( K_{r, \rho}, r \in \mathbb{Q}_M, \rho \in R_M \)
- \( D_\rho, \rho \in R_M \).
- \( L_{Q_p}^D \) model \( M \) is same as \( L_{Q_p} \) model and

\[ \mathcal{M} \models D_{r, \rho} \alpha \text{ iff } |\mu([\alpha]) - \mu([\beta])|_p \leq \rho \]
Axiomatization

1. Substitutional instances of tautologies;
2. $K_{r,\rho}\alpha \Rightarrow K_{r,\rho'}\alpha$, whenever $\rho' \geq \rho$;
3. $K_{r_1,\rho_1}\alpha \land K_{r_2,\rho_2}\beta \land K_{0,0}(\alpha \land \beta) \Rightarrow K_{r_1+r_2,\max\{\rho_1,\rho_2\}}(\alpha \lor \beta)$;
4. $K_{r_1,\rho_1}\alpha \Rightarrow \neg K_{r_2,\rho_2}\alpha$, if $|r_1 - r_2| > \max\{\rho_1,\rho_2\}$;
5. $K_{r_1,\rho}\alpha \Rightarrow K_{r_2,\rho}\alpha$, if $|r_1 - r_2| \leq \rho$;
6. $K_{r,\rho_1}\alpha \land D_{\rho_2}\alpha, \beta \Rightarrow K_{r,\max\{\rho_1,\rho_2\}}\beta$;
7. $K_{r,\rho}\alpha \land K_{r,\rho}\beta \Rightarrow D_{\rho}\alpha, \beta$;
1. From $A$ and $A \Rightarrow B$ infer $B$. Here $A$ and $B$ are either both propositional, or both probabilistic formulas;

2. From $\alpha$ infer $K_{c,0}\alpha$, $c \in \mathbb{Q}_M$, $c \neq 0$

3. From $\alpha \Rightarrow \bot$ infer $K_{0,0}\alpha$;

4. If $n \in \mathbb{N}$, from $\varphi \Rightarrow \neg K_{r, p^M-n}\alpha$ for every $r \in \mathbb{Q}_M$, infer $\varphi \Rightarrow \bot$.

5. If $r \in \mathbb{Q}_M$, from $\varphi \Rightarrow K_{r, p^M-n}\alpha$ for every $n \in \mathbb{N}$, infer $\varphi \Rightarrow K_{r,0}\alpha$.

6. From $\varphi \Rightarrow D_{p^M-n}\alpha, \beta$ for every $n \in \mathbb{N}$, infer $\varphi \Rightarrow D_{0}\alpha, \beta$.

7. From $\alpha \iff \beta$ infer $K_{r,\rho}\alpha \Rightarrow K_{r,\rho}\beta$. 
References

