A p-adic probability logic

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Outline

1. Probability logic
2. L_{Q_p}
3. $CPL^{fin}_{\mathbb{Q}_p}$, $CPL_{\mathbb{Z}_p}$
4. $L^D_{Q_p}$

Basic concepts

- Weighted logics
- Predicate and operator logics
- Completeness
- Decidability and complexity

Issues related to range

- Noncompactness: $\{Pr(\alpha) \neq r \ : \ r \in X\}$ is finitely satisfiable for any infinite X;
- Completeness fails for uncountable languages.

p-adic probability space

Definition

Let $r = p^n$ ($n \in \mathbb{Z}$), $K[0, r] = \{x \in \mathbb{Q}_p \ : \ \|x\|_p \leqslant r\}$.
A p–adic r–probability space is any structure (W, H, μ) such that $W \neq \emptyset$, $H \leqslant P(W)$ and $\mu : H \longrightarrow K[0, r]$ is additive.

- *Boundedness condition*: for every $A \in H$

$$\|A\| = sup\{|\mu(B)|_p : B \in A, B \subset A\} < \infty$$

Basic p-adic probability logic L_{Q_p}

Enriches propositional calculus with the operators:

$$K_{r,\rho}\alpha$$

with the intended meaning:

"the probability of α belongs to the p-adic ball with the center r and the radius ρ"

Formal language

Let p be a fixed prime and $M \in \mathbb{N}$ be an arbitrary large but fixed positive integer. We introduce the following sets:

1. $\mathbb{Q}_M = \{r \in \mathbb{Q} : |r|_p \leqslant p^M\}$;
2. $\mathbb{Z}_M = \mathbb{Z}^- \cup \{0, 1, 2, \ldots, M\}$, where $\mathbb{Z}^-$ denotes the set of all negative integers;
3. $R_M = \{p^{M-n} : n \in \mathbb{N}\} \cup \{0\} = \{p^n : n \in \mathbb{Z}_M\} \cup \{0\}$.

Formal language

- $Var = \{p_1, p_n, \ldots\}$: countable set of propositional letters, connectives $\neg$ and $\wedge$;
- Probability operators $K_{r,\rho}$, $r \in \mathbb{Q}_M$, $\rho \in R_M$;
- For_{Cl}-the set of classical propositional formulas over Var;

Definition

The set For_P of all probabilistic formulas is defined as the least set satisfying the following conditions:

- If $\alpha, \beta \in For_{Cl}$, $r \in \mathbb{Q}_M, \rho \in R$ then $K_{r,\rho}\alpha, \beta$ is probabilistic formula.
- If φ, ϕ are probabilistic formulas, then $(\neg\varphi)$, $(\varphi \wedge \phi)$ are probabilistic formulas.

Semantics

Definition

An L_{Q_p}-model is a structure $\mathcal{M} = \langle W, H, \mu, v \rangle$ where:

- W is a nonempty set of elements called worlds;
- H is an algebra of subsets of W;
- $\mu : H \longrightarrow K[0, p^M]$ is a measure (additive function) such that $\mu(W) = 1$;
- $v : W \times Var \longrightarrow \{true, false\}$ is a valuation which associated with every world $w \in W$ a truth assignment $v(w, \cdot)$ on propositional letters; the valuation $v(w, \cdot)$ is extended to classical propositional formulas as usual.

If M is an L_{Q_p}-model, by $[\alpha]_M$ we denote the set w such that $v(w, \alpha) = true$.

Satisfiability

Definition

Let $\mathcal{M} = \langle W, H, \mu, v \rangle$ be an L_{Q_p}-model. The satisfiability relation is inductively defined as follows:

- If $\alpha \in For_{Cl}$ then $\mathcal{M} \models \alpha$ iff $v(w, \alpha) = true$ for all $w \in W$;
- If $\alpha \in For_{Cl}$ then $\mathcal{M} \models K_{r,\rho}\alpha$ iff $|\mu([\alpha]) - r|_p \leqslant \rho$;
- If $\varphi \in For_P$ then $\mathcal{M} \models \neg\varphi$ iff it is not $\mathcal{M} \models \varphi$;
- If $\varphi, \psi \in For_P$ then $\mathcal{M} \models \varphi \wedge \psi$ iff $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$.

Axiomatization

The axiom system $AX_{L_{Q_p}}$ of the logic L_{Q_p} contains the following axioms and inference rules:

Axioms

- A1 Substitutional instances of tautologies;
- A2 $K_{r,\rho}\alpha \Rightarrow K_{r,\rho'}\alpha$, whenever $\rho' \geqslant \rho$;
- A3 $K_{r_1,\rho_1}\alpha \wedge K_{r_2,\rho_2}\beta \wedge K_{0,0}(\alpha \wedge \beta) \Rightarrow K_{r_1+r_2,\max(\rho_1,\rho_2)}(\alpha \vee \beta)$;
- A4 $K_{r_1,\rho_1}\alpha \Rightarrow \neg K_{r_2,\rho_2}\alpha$, if $|r_1 - r_2|_p > \max(\rho_1, \rho_2)$;
- A5 $K_{r_1,\rho}\alpha \Rightarrow K_{r_2,\rho}\alpha$, if $|r_1 - r_2|_p \leqslant \rho$;

Inference rules

- R1 Modus ponens: From Φ and $\Phi \Rightarrow \Psi$ infer Ψ;
- R2 Necessitation (a): from α infer $K_{1,0}\alpha$;
- R3 Necessitation (b): from $\neg\alpha$ infer $K_{0,0}\alpha$;
- R4 Coherence: From $\alpha \Leftrightarrow \beta$ infer $K_{r,\rho}\alpha \Rightarrow K_{r,\rho}\beta$;
- R5 Range: From $\phi \Rightarrow \neg K_{r,p^{M-n}}\alpha$ for all $r \in \mathbb{Q}_M$, infer $\neg\phi$;
- R6 Convergence: From $\phi \Rightarrow_K r, \rho\alpha$ for all $\rho \in R_M$, infer $\phi \Rightarrow K_{r,0}\alpha$.

Soundness

Theorem

The axiomatic system $AX_{L_{Q_p}}$ is sound with respect to the class of L_{Q_p}-models

Theorem

(Deduction theorem) Let T be a set of formulas and A and B both classical or both propositional formulas. Then, $T, A \vdash B$ implies $T \vdash A \Rightarrow B$.

Completeness

Theorem

Every consistent set of formulas T can be extended to a maximal consistent set.

Completeness

- T^* maximal consistent set of formulas;
- T^* satisfies: for every formula $\alpha \in For_{Cl}$ and every $m \in \mathbb{N}$ there are countably many $r \in \mathbb{Q}_M$ such that $K_{r,p^{M-m}}\alpha \in T^*$;
- For every $\alpha \in For_{Cl}$ we define sequence r_m in the following way:
  - For every $m \in \mathbb{N}$ we arbitrarily choose r such that $K_{r,p^{M-m}}\alpha \in T^*$ and this r will be m-th number of the sequence,i.e., $r_m = r$.
- For $\alpha \in For_{Cl}$ we obtain sequence $r(\alpha) = r_0, r_1, \ldots,$ where $K_{r_j,p^{M-j}}\alpha \in T^*$.

Lemma

Let $r(\alpha)$ be defined as above.Then $r(\alpha)$ is a Cauchy sequence with respect to the p-adic norm. Moreover, the limit of $r(\alpha)$ does not depend on the choice of r_k's.

Completeness-canonical model

Let $M_{T^*} = \langle W, H, \mu, v \rangle$, where:

- T is consistent set, $\overline{T}$ is set of all classical consequences of T
- $W = \{w | v(w, \alpha) = \mathit{true}$ for every $\alpha \in \overline{T}\}$ contains all classical propositional interpretations that satisfy $\overline{T}$,
- $H = \{[\alpha] : \alpha \in For_{Cl}\}$
- $\mu : H \to \mathbb{Q}_p$: Let $r(\alpha) = (r_n)_{n \in \mathbb{N}}$. Then

$$\mu([\alpha]) = \begin{cases} r & \text{if } K_{r,0}\alpha \in T^* \\ \lim^p_{n\to\infty} r_n & \text{otherwise} \end{cases}$$

- for every world w and every $p \in Var$, $v(w, p) = \mathit{true}$ iff $w \models p$.

Completeness, Decidability

Theorem
A set of formulas T is consistent iff has an L_{Q_p}-model.

Theorem
The satisfiability problem for L_{Q_p} formulas is decidable.

Satisfying boundedness condition

- *boundedness condition*: If F is a field of subsets of some set Ω then, for every $A \in F$

$$sup\{|\mu(B)|_p : B \in F, B \subset A\} < \infty$$

- This condition can be ensured by reducing the range of probabilities to an arbitrarily large (but fixed) ball $K[0, p^M]$, where M is some fixed integer.
- $P(A|B) \cdot P(B) = P(A \wedge B)$; $K[0, p^M]$ is not closed for multiplication!

Satisfying boundedness condition

- We might proceed in two ways.
  1. Using unit ball $K[0,1]$ as a range of probability, since it closed for multiplication-$CPL_{\mathbb{Z}_p}$
  2. To built formulas from the finite set of propositional letters, but to retain $\mathbb{Q}_p$ as a range of probability-$CPL_{\mathbb{Q}_p}^{fin}$.
     In this way we compute supremum of finitely many numbers of the form p^n, $n \in \mathbb{Z}$, which is again a finite number, precisely:

  - *boundedness condition* : For every $\alpha \in For_{Cl}$ in the model $M = \langle W, H, \mu, v \rangle$

$$sup\{|\mu([\beta])|_p : [\beta] \in H, [\beta] \subset [\alpha]\} < \infty$$

  - For every $\alpha \in For_{Cl}$ there exist finitely many logically inequivalent formulas β such that $\beta \Rightarrow \alpha$ is tautology, i.e., such that $[\beta] \subset [\alpha]$.

Formal language

$CK_{r,\rho}\alpha, \beta$: "the conditional probability of α given β is in the p-adic ball with the center r and the radius ρ"

- $CPL^{fin}_{\mathbb{Q}_p}$: $Var = \{p_1, \ldots, p_n\}$;
- $CPL_{\mathbb{Z}_p}$
  - $\mathbb{Q}_1 = \{r \in \mathbb{Q} \mid |r|_p \leq 1\}$,
  - $R = \{p^{-n} \mid n \in \mathbb{N}\} \cup \{0\}$,
  - $(CK_{r,\rho})_{r \in \mathbb{Q}_1, \rho \in R}$.
- $CPL^{fin}_{\mathbb{Q}_p}$:
  - $R_1 = \{p^n | n \in \mathbb{Z}\} \cup \{0\}$
  - $CK_{r,\rho}$ $r \in \mathbb{Q}$, $\rho \in R_1$

Semantics

Definition

A CPL_{Z_p},($CPL^{fin}_{Q_p}$)-model is a structure $M = \langle W, H, \mu, v \rangle$ where:

- W is a nonempty set of elements called worlds.
- H is an algebra of subsets of W.
- $\mu : H \to \mathbb{Z}_p (\mu : H \to \mathbb{Q}_p)$ is a measure (additive function) such that $\mu(W) = 1$.
- $v : W \times Var \to \{true, false\}$ is a valuation which associates with every world $w \in W$ a truth assignment $v(w, \cdot)$ on propositional letters; the valuation $v(w, \cdot)$ is extended to classical propositional formulas as usual.

Satisfiability

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Definition

Let $M = \langle W, H, \mu, v \rangle$ be an $CPL_{\mathbb{Z}_p}$, ($CPL_{\mathbb{Q}_p}^{fin}$)-model. The satisfiability relation is inductively defined as follows:

- If $\alpha \in For_{Cl}$ then $M \models \alpha$ iff $v(w, \alpha) = true$ for every $w \in W$.
- If $\alpha, \beta \in For_{Cl}$ then $M \models CK_{r,\rho}\alpha, \beta$ iff:
  - $\mu([\beta]) = 0$ and $|r - 1|_p \leq \rho$ or
  - $\mu([\beta]) \neq 0$ and $|\frac{\mu([\alpha \wedge \beta])}{\mu([\beta])} - r|_p \leq \rho$.
- If $\varphi \in For_P$, then $M \models \neg\varphi$ iff it is not $M \models \varphi$.
- If $\varphi, \psi \in For_P$ then $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$.

Satisfiability

- $M \models CK_{r,\rho}\alpha, \top$ iff $|\mu([\alpha]) - r|_p \leq \rho$.
- Conditional probability, $P(\alpha|T)$ comes to standard probability $P(\alpha)$.
- $CK_{r,\rho}\alpha, \top$ will be denoted by $K_{r,\rho}\alpha$.

Axiomatization

1. Substitutional instances of tautologies .
2. $K_{r_1,\rho_1}\alpha \wedge K_{r_2,\rho_2}\beta \wedge K_{0,0}(\alpha \wedge \beta) \Rightarrow K_{r_1+r_2,max(\rho_1,\rho_2)}(\alpha \vee \beta)$.
3. $CK_{r,\rho}\alpha, \beta \Rightarrow CK_{r,\rho'}\alpha, \beta, \;\; \rho' \geq \rho$
4. $CK_{r_1,\rho_1}\alpha, \beta \Rightarrow \neg CK_{r_2,\rho_2}\alpha, \beta$, if $|r_1 - r_2|_p > max(\rho_1, \rho_2)$,
5. $CK_{r_1,\rho}\alpha, \beta \Rightarrow CK_{r_2,\rho}\alpha, \beta$, if $|r_1 - r_2|_p \leq \rho$

$CPL_{\mathbb{Z}_p}$ $K_{r_1 r_2,\rho_1}(\alpha \wedge \beta) \wedge K_{r_2,\rho_2}\beta \Rightarrow CK_{r_1,\frac{\max\{\rho_1,\rho_2\}}{|r_2|_p}}\alpha, \beta$ whenever $r_2 \neq 0$, $|r_1|_p \leq 1$, $|r_2|_p > \rho_2$;

$CPL_{\mathbb{Q}_p}^{fin}$ $K_{r_1 r_2,\rho_1}(\alpha \wedge \beta) \wedge K_{r_2,\rho_2}\beta \Rightarrow CK_{r_1,\frac{max\{\rho_1,|r_1|_p\cdot\rho_2\}}{|r_2|_p}}\alpha, \beta$ $r_2 \neq 0$, $|r_2|_p > \rho_2$

7. $CK_{r,\rho}\alpha, \beta \wedge K_{r_1,\rho_1}\beta \Rightarrow K_{r\cdot r_1,max\{|r_1|_p\cdot\rho,|r|_p\cdot\rho_1\}}\alpha \wedge \beta$, if $r_1 \neq 0$, $|r_1|_p > \rho_1$.
8. $K_{0,0}\beta \wedge K_{r,\rho}(\alpha \wedge \beta) \Rightarrow CK_{1,0}\alpha, \beta$

Inference rules

1. From A and $A \Rightarrow B$ infer B. Here A and B are both propositional, or both probabilistic formulas.
2. From α infer $K_{1,0}\alpha$
3. If $n \in \mathbb{N}$, from $\varphi \Rightarrow \neg K_{r,p^{-n}}\alpha$ for every $r \in \mathbb{Q}$, infer $\varphi \Rightarrow \perp$.
4. From $\alpha \Rightarrow \perp$, infer $K_{0,0}\alpha$
5. If $r \in \mathbb{Q}$, from $\varphi \Rightarrow CK_{r,p^{-n}}\alpha, \beta$ for every $n \in \mathbb{N}$, infer $\varphi \Rightarrow CK_{r,0}\alpha, \beta$.
6. From $\alpha \Leftrightarrow \beta$ infer $(K_{r,\rho}\alpha \Leftrightarrow K_{r,\rho}\beta)$.

Formal language, Semantics

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$D_\rho\alpha, \beta$: "the p-adic distance between the probabilities of α and β is less than or equal to ρ"

- $K_{r,\rho}$, $r \in \mathbb{Q}_M$, $\rho \in R_M$
- D_ρ, $\rho \in R_M$.
- $L^D_{Q_p}$ model $\mathcal{M}$ is same as L_{Q_p} model and

$$\mathcal{M} \models D_{r,\rho}\alpha \text{ iff } |\mu([\alpha]) - \mu([\beta])|_p \leqslant \rho$$

Axiomatization

1. Substitutional instances of tautologies;
2. $K_{r,\rho}\alpha \Rightarrow K_{r,\rho'}\alpha$, whenever $\rho' \geq \rho$;
3. $K_{r_1,\rho_1}\alpha \wedge K_{r_2,\rho_2}\beta \wedge K_{0,0}(\alpha \wedge \beta) \Rightarrow K_{r_1+r_2,\max\{\rho_1,\rho_2\}}(\alpha \vee \beta)$;
4. $K_{r_1,\rho_1}\alpha \Rightarrow \neg K_{r_2,\rho_2}\alpha$, if $|r_1 - r_2|_p > \max\{\rho_1, \rho_2\}$;
5. $K_{r_1,\rho}\alpha \Rightarrow K_{r_2,\rho}\alpha$, if $|r_1 - r_2|_p \leq \rho$;
6. $K_{r,\rho_1}\alpha \wedge D_{\rho_2}\alpha, \beta \Rightarrow K_{r,\max\{\rho_1,\rho_2\}}\beta$;
7. $K_{r,\rho}\alpha \wedge K_{r,\rho}\beta \Rightarrow D_{\rho}\alpha, \beta$;

1. From A and $A \Rightarrow B$ infer B. Here A and B are either both propositional, or both probabilistic formulas;
2. From α infer $K_{c,0}\alpha$, $c \in \mathbb{Q}_M, c \neq 0$
3. From $\alpha \Rightarrow \perp$ infer $K_{0,0}\alpha$;
4. If $n \in \mathbb{N}$, from $\varphi \Rightarrow \neg K_{r,p^{M-n}}\alpha$ for every $r \in \mathbb{Q}_M$, infer $\varphi \Rightarrow \perp$.
5. If $r \in \mathbb{Q}_M$, from $\varphi \Rightarrow K_{r,p^{M-n}}\alpha$ for every $n \in \mathbb{N}$, infer $\varphi \Rightarrow K_{r,0}\alpha$.
6. From $\varphi \Rightarrow D_{p^{M-n}}\alpha, \beta$ for every $n \in \mathbb{N}$, infer $\varphi \Rightarrow D_0\alpha, \beta$.
7. From $\alpha \Leftrightarrow \beta$ infer $K_{r,\rho}\alpha \Rightarrow K_{r,\rho}\beta$.

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