Fuglede Conjecture on $\mathbb{Q}_p$

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International Conference on p-adic Mathematical Physics and Its Application, Belgrade, Serbia

September 8, 2015

Outline

Question

Preliminaries

Fuglede Conjecture for Compact Open Set in $\mathbb{Q}_p$
- Geometry of Compact Open Spectral Set
- Spectra and Translates

Fuglede Conjecture on $\mathbb{Q}_p$

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Spectral set

- A character of a local compact abelian group G is a group homomorphism $\chi : G \to S^1$, i.e. $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ and $\chi(0) = 1$.
- $\hat{G}$: the dual group which consists of all the characters of G.
- A subset $\Omega \subset G$ of finite Haar Measure is said to be spectral if there exists a set $\Lambda \subset \hat{G}$ which form a Hilbert basis of $L^2(\Omega)$. The set Λ is called a spectrum of Ω and (Ω, Λ) is called a spectral pair.

Tiling

- For an element $x \in G$,

$$\Omega + x := \{y + x \in G : y \in \Omega\}.$$

- We say that the set Ω tiles G by translation, if there exists a set $T \subset G$ such that $\{\Omega + t : t \in T\}$ forms a partition a.e. of G, equivalently,

$$\sum_{t \in T} 1_{\Omega}(x - t) = 1, \qquad a.e. \;\; x \in G.$$

The set T is called a translate of Ω and (Ω, T) is called a tiling pair.

Spectral set conjecture

Question

Ω is a spectral set if and only if it tiles G?

- The case $G = \mathbb{Z}$ is open. The case $G = \mathbb{Z}/p^n\mathbb{Z}$ is true.
- The case $G = \mathbb{R}^d$ is the famous Fuglede conjecture.
  - The conjecture is not true for $d \geq 3$(both direction). For $d = 5$, Terence Tao gave a counterexample in 2004. For d $=$ 3, 4, Matolsci and Kolountzakis (2006) were able to use Tao's idea to give counterexamples and showed that both implications of Fuglede's fail.
  - It is still open for $d = 1, 2$. It is true for convex planer sets (Katz and Tao). For $d = 1$, Laba, Lagarias, Wang....
- How about the case $G = \mathbb{Q}_p^d$? We confirm the conjecture when $d = 1$.

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Notation

- $\mathbb{Q}_p$: the filed of p-adic numbers.
- $\mathbb{Z}_p$: the ring of p-adic integers.
- $\mathfrak{m}$ or dx: the Haar measure on $\mathbb{Q}_p$ such that $\mathfrak{m}(\mathbb{Z}_p) = 1$.
- Any $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{n=v_p(x)} a_n p^n \ (v_p(x) \in \mathbb{Z}, a_n \in \{0, 1, \cdots, p-1\} \text{ and } a_{v_p(x)} \neq 0).$$

  The fractional part $\{x\}$ of x is defined to be $\sum_{n=v_p(x)}^{-1} a_n p^n$
- Pontryagin duality theorem: the dual group $\widehat{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is isomorphic to $\mathbb{Q}_p$.

Dual group of $\mathbb{Q}_p$

- Fix the following character $\chi \in \widehat{\mathbb{Q}_p}$:

$$\chi(x) = e^{2\pi i \{x\}}.$$

Notice: χ is a local constant function($\chi(x) = 1$, if $x \in \mathbb{Z}_p$).

- For any $y \in \mathbb{Q}_p$, we define

$$\chi_y(x) = \chi(yx).$$

The map $y \mapsto \chi_y$ from $\mathbb{Q}_p$ onto $\widehat{\mathbb{Q}_p}$ is an isomorphism.

Fourier transform

For $f \in L^1(\mathbb{Q}_p)$, the Fourier transform of f is defined to be

$$\widehat{f}(y) = \int_{\mathbb{Q}_p} f(x)\overline{\chi}_y(x)dx.$$

Example

- $\widehat{1_{B(0,p^\gamma)}}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi)$
- Let $\Omega = \bigsqcup_{j=1} B(c_j, p^\gamma)$. Then $\widehat{1_\Omega}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi) \sum_j \chi(-c_j\xi)$.

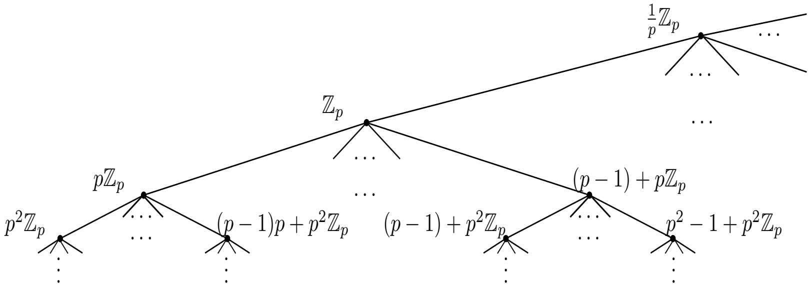
Lemma (A criterion of spectral set)

A Borel set Ω of finite haar measure is a spectral set with Λ as a spectrum iff

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \sum_{\lambda \in \Lambda} |\widehat{1_\Omega}(\lambda - \xi)|^2 = \mathfrak{m}(\Omega)^2.$$

Tree structure of $\mathbb{Q}_p$

Vertices $\mathcal{T}$: balls in $\mathbb{Q}_p$.
Edges $\mathcal{E}$: pairs $(B', B) \in \mathcal{T} \times \mathcal{T}$ such that $B' \subset B$, $\mathfrak{m}(B) = p\mathfrak{m}(B')$, denote by $B' \prec B$.



Bounded open sets in $\mathbb{Q}_p$

Any bounded open set O of $\mathbb{Q}_p$ can be described by a subtree $(\mathcal{T}_O, \mathcal{E}_O)$ of $(\mathcal{T}, \mathcal{E})$.

- Let B^* be the smallest ball containing O, which will be the root of the tree. For any given ball B contained in O, there is a unique sequence of balls $B_0, B_1, \cdots, B_r$ such that

$$B = B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_r = B^*.$$

- The set of vertices $\mathcal{T}_O$ is composed of all such balls $B_0, B_1, \cdots, B_r$ for all possible balls B contained in O.
- The set of edges $\mathcal{E}_O$ is composed of all edges $B_i \prec B_{i+1}$ as above.

Comapact open set in $\mathbb{Q}_p$

Any compact open set can be described by a finite tree, because a compact open set is a disjoint finite union of balls of same size. In this case, as in the above construction of subtree we only consider these balls of same size as B.

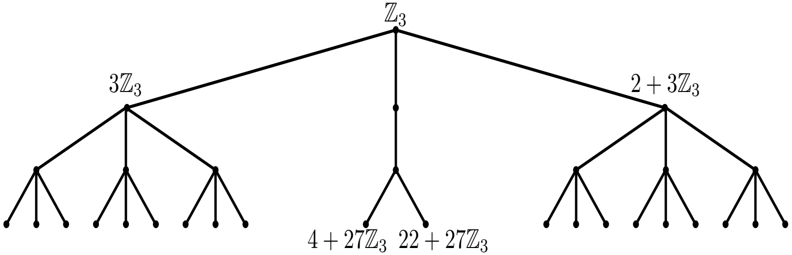


Figure: $\Omega = 3\mathbb{Z}_3 \sqcup (2+3\mathbb{Z}_3) \sqcup (4+27\mathbb{Z}_3) \sqcup (22+27\mathbb{Z}_3)$.

p-homogenous subsets in $\mathbb{Q}_p$

- A subtree $(\mathcal{T}', \mathcal{E}')$ is said to be homogeneous if the number of descendants of $B \in \mathcal{T}'$ depends only on $|B|$. If this number is either 1 or p, we call $(\mathcal{T}', \mathcal{E}')$ a p-homogeneous tree.
- An bounded open set is said to be homogeneous (resp. p-homogeneous) if the corresponding tree is homogeneous (resp. p-homogeneous).
- A bounded open p-homogenous set must be compact.

p-homogenous subsets in $\mathbb{Q}_p$

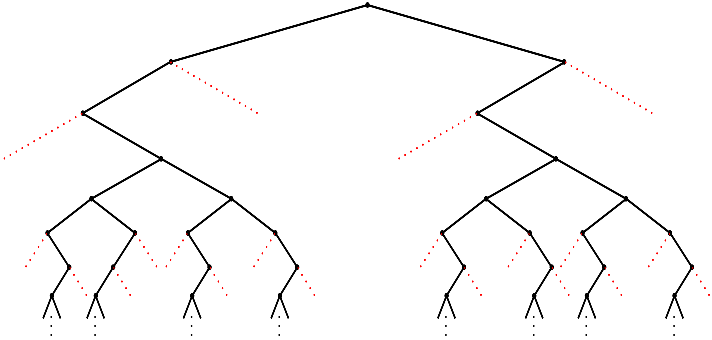


Figure: A 2-homogenous tree

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Tree structure of $\mathbb{Z}/p^\gamma\mathbb{Z}$

We identify $\mathbb{Z}/p^\gamma\mathbb{Z} = \{0, 1, \cdots, p^\gamma - 1\}$ with $\{0, 1, 2, \cdots p - 1\}^\gamma$ which is considered as a finite tree, denoted by $\mathcal{T}^{(\gamma)}$.

- Vertices $\mathcal{T}^{(\gamma)}$: consists of the disjoint union of the sets $\mathbb{Z}/p^n\mathbb{Z}, 0 \leq n \leq \gamma$. Each vertex, except the root of the tree, is identified with a sequence $t_0 t_1 \cdots t_n$ with $0 \leq n \leq \gamma$ and $t_i \in \{0, 1, \cdots, p - 1\}$.
- Edges: consists of pairs $(x, y) \in \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n+1}\mathbb{Z}$ with $x \equiv y \mod p^n$, where $0 \leq n \leq \gamma - 1$.

For example, each point t of $\mathbb{Z}/p^\gamma\mathbb{Z}$ is identified with $t_0 t_1 \cdots t_{\gamma-1}$, which is a boundary point of the tree.

Tree structure of $\mathbb{Z}/p^\gamma\mathbb{Z}$

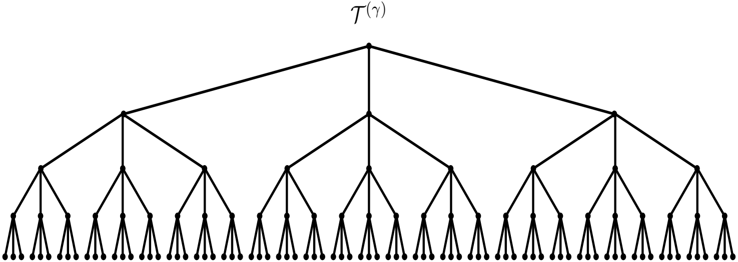

Figure: The set $\mathbb{Z}/3^4\mathbb{Z} = \{0, 1, 2, \cdots, 80\}$ is considered as a tree $\mathcal{T}^{(4)}$.

p-homogenous subsets $\mathbb{Z}/p^\gamma\mathbb{Z}$

Each subset $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ will determine a subtree of $\mathcal{T}^{(\gamma)}$, denoted by $\mathcal{T}_C$, which consists of the paths from the root to the points in C. For each $0 \le n \le \gamma$, denote $C_{\text{mod } p^n} := \{x \in \mathbb{Z}/p^n\mathbb{Z} : \exists\ y \in C,$ such that $x = y \bmod p^n\}$.

- Vertices $\mathcal{T}_C$: consists of the disjoint union of the sets $C_{\text{mod } p^n}, 0 \le n \le \gamma$.
- Edges: consists of pairs $(x, y) \in C_{\text{mod } p^n} \times C_{\text{mod } p^{n+1}}$ with $x \equiv y \mod p^n$, where $0 \le n \le \gamma - 1$.

The set C is called a p-homogenous subsets of $\mathbb{Z}/p^\gamma\mathbb{Z}$ iff the corresponding tree $\mathcal{T}_C$ is p-homogenous.

p-homogenous subsets $\mathbb{Z}/p^{\gamma}\mathbb{Z}$

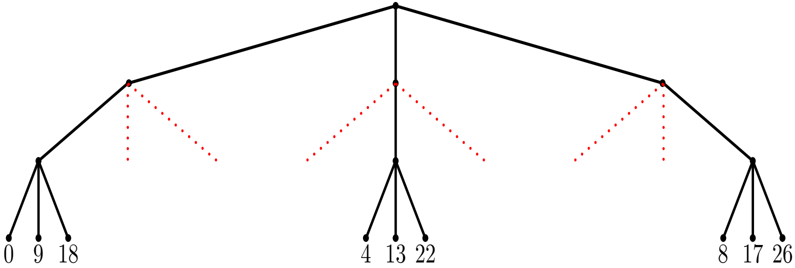

Figure: For $p = 3, \gamma = 2$, the tree p-homogeneous tree determined by $\{0, 4, 8, 9, 13, 17, 18, 22, 26\}$.

Spectral sets and tiles in $\mathbb{Z}/p^\gamma\mathbb{Z}$

Recall that the Fourier transform of a function f defined on $\mathbb{Z}/p^\gamma\mathbb{Z}$ is defined as

$$\widehat{f}(k) = \sum_{x\in\mathbb{Z}/p^\gamma\mathbb{Z}} f(x)e^{-\frac{2\pi ikx}{p^\gamma}}, (\forall k \in \mathbb{Z}/p^\gamma\mathbb{Z}).$$

Theorem (Fan-F-Shi) Let $C \in \mathbb{Z}/p^\gamma\mathbb{Z}$. The following statements are equivalent.

(1) C is p-homogenous.

(2) For any $1 \leq i \leq \gamma, \sharp(C_{\text{mod } p^i}) = p^{k_i}$, for some $k_i \in \mathbb{N}$.

(3) There exists a subset $I \subset \mathbb{N}$ such that $\sharp I = \log_p(\sharp C)$ and $\widehat{1_C}(p^\ell) = 0$ for $\ell \in I$.

(4) C tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$.

(5) C is a spectral set in $\mathbb{Z}/p^\gamma\mathbb{Z}$.

Spectral sets and tiles in $\mathbb{Z}/p^\gamma\mathbb{Z}$

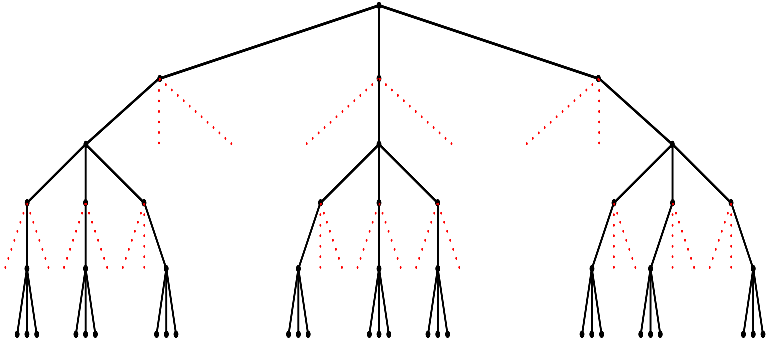

Figure: For $p = 3$, tree $\mathcal{T}_{I,J}$ with $\gamma = 5, I = \{0, 2, 4\}, J = \{1, 3\}$.

Compact open spectral sets $\mathbb{Q}_p$

W. l. o. g, we assume that Ω is of the form

$$\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \cdots, p^\gamma - 1\}$.
Theorem (Fan-F-Shi) The following are equivalent.

(1) $\mathcal{T}_C$ is a p-homogenous tree.

(2) Ω is p-homogenous.

(3) Ω tiles $\mathbb{Q}_p$.

(4) Ω is a spectral set in $\mathbb{Q}_p$.

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Spectra and Translates

For a subset $\Omega \subset \mathbb{Q}_p$, the set of admissible p-orders of Ω:

$$I_\Omega := \{i \in \mathbb{Z}, \exists\ x, y \in \Omega \text{ such that } |x-y|_p = p^{-i}\}.$$

Remark

- For compact open Ω, $\exists\ \gamma \in \mathbb{Z}$ such that $i \in I_\Omega$ if $i \geq \gamma$.
- For p-homogenous compact open set Ω, an integer $i \in I_\Omega$ iff the balls of radius p^{-i} in the tree $\mathcal{T}_\Omega$ has p descendants.
- For two p-homogenous compact open set Ω and Ω', $I_\Omega = I_{\Omega'}$ iff $\exists$ isometric transformation $f : \mathbb{Q}_p \to \mathbb{Q}_p$ such that $f(\Omega) = \Omega'$.

Spectra and Translates

For a discrete subset E in $\mathbb{Q}_p$, we call E a uniformly discrete set if I_E is upper bounded. For each integer n, denote

$$I_E^n := \{i \in I_E : i \geq -n\}.$$

A uniformly discrete set E is called p-homogenous discrete if

$$\sharp(E \cap B(a, p^n)) = p^{\sharp I_E^n} \text{ or } 0, \quad \forall a \in \mathbb{Q}_p.$$

Remark

Let E and E' be two p-homogenous discrete subset in $\mathbb{Q}_p$. Then $I_E = I_{E'}$ iff $\exists$ isometric transformation $f : \mathbb{Q}_p \to \mathbb{Q}_p$ such that $f(E) = E'$.

Spectra and Translates

Theorem (Fan-F-Shi)

Let Ω be a p-homogenous compact open set. Assume that (Ω, Λ) is a spectral pair and (Ω, T) is a tiling pair.

- *Each spectrum Λ is p-homogenous discrete with $I_\Lambda = -(I_\Omega + 1)$.*
- *Each translate T is p-homogenous discrete with $I_T = \mathbb{Z} \setminus I_\Omega$.*

Spectra and Translates

Theorem (Fan-F-Shi)

Let Ω be a p-homogeneous compact open set in $\mathbb{Q}_p$.

- *Subject to an isometric bijection,*

$$\Lambda = \sum_{i \in I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^{-i-1} \subset \mathbb{Q}_p$$

is the unique spectrum of Ω.

- *Subject to an isometric bijection,*

$$T = \sum_{i \notin I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^{i} \subset \mathbb{Q}_p$$

is the unique tiling complement of Ω.

Spectra and Translates

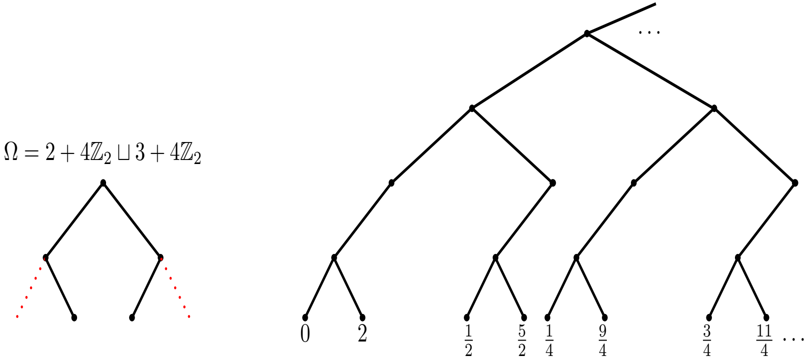

Figure: Consider the translate T of Ω as an infinite tree and the points of T are the boundary points of the tree.

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A set $\Omega \subset \mathbb{Q}_p$ is called an almost compact open set, if $\exists$ compact open $\Omega' \subset \mathbb{Q}_p$ such that

$$\mathfrak{m}(\Omega \setminus \Omega') = \mathfrak{m}(\Omega' \setminus \Omega) = 0.$$

Theorem

A Borel set $\Omega \in \mathbb{Q}_p$ is a spectral set if and only if it tiles $\mathbb{Q}_p$. Moreover, Ω is an almost compact open set .

Theorem

A subset E of $\mathbb{Q}_p$ is a spectrum iff it is a translate. Moreover, it is a p-homogenous discrete subset of $\mathbb{Q}_p$.

Idea of Proof

- Calculate the densities of spectrum Λ and translate T.
- Consider μ_Λ and μ_T as distributions in $\mathbb{Q}_p$ (continuous linear functionals on space of the local constant function with compact support) . Fourier analysis on $\mathbb{Q}_p$ (See Ableverio Khrennikov and Shelkovich' s book 2010, Vladimirov and Volovich and Zelenov' s book 1994).
- Analyze the zeros of $\widehat{\mu_\Lambda}$ and $\widehat{\mu_T}$. Obtain the structure of Λ and T(p-homogenous discrete).

The end

Thank you!