Fuglede Conjecture on $\mathbb{Q}_p$

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Outline

Question

Preliminaries

Fuglede Conjecture for Compact Open Set in $\mathbb{Q}_p$
  Geometry of Compact Open Spectral Set
  Spectra and Translates

Fuglede Conjecture on $\mathbb{Q}_p$
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Fuglede Conjecture on $\mathbb{Q}_p$
Spectral set

- A character of a local compact abelian group $G$ is a group homomorphism $\chi : G \to S^1$, i.e. $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ and $\chi(0) = 1$.
- $\hat{G}$: the dual group which consists of all the characters of $G$.
- A subset $\Omega \subset G$ of finite Haar Measure is said to be spectral if there exists a set $\Lambda \subset \hat{G}$ which form a Hilbert basis of $L^2(\Omega)$. The set $\Lambda$ is called a spectrum of $\Omega$ and $(\Omega, \Lambda)$ is called a spectral pair.
For an element $x \in G$,

$$\Omega + x := \{ y + x \in G : y \in \Omega \}.$$ 

We say that the set $\Omega$ tiles $G$ by translation, if there exists a set $T \subset G$ such that $\{ \Omega + t : t \in T \}$ forms a partition a.e. of $G$, equivalently,

$$\sum_{t \in T} 1_{\Omega}(x - t) = 1, \quad a.e. \quad x \in G.$$ 

The set $T$ is called a translate of $\Omega$ and $(\Omega, T)$ is called a tiling pair.
Spectral set conjecture

Question

Ω is a spectral set if and only if it tiles G?

- The case $G = \mathbb{Z}$ is open. The case $G = \mathbb{Z}/p^n\mathbb{Z}$ is true.
- The case $G = \mathbb{R}^d$ is the famous Fuglede conjecture.
  - The conjecture is not true for $d \geq 3$ (both direction). For $d = 5$, Terence Tao gave a counterexample in 2004. For $d = 3, 4$, Matolsci and Kolountzakis (2006) were able to use Tao’s idea to give counterexamples and showed that both implications of Fuglede’s fail.
  - It is still open for $d = 1, 2$. It is true for convex planer sets (Katz and Tao). For $d = 1$, Laba, Lagarias, Wang….
- How about the case $G = \mathbb{Q}_p^d$? We confirm the conjecture when $d = 1$. 
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Fuglede Conjecture on $\mathbb{Q}_p$
Notation

- $\mathbb{Q}_p$: the field of $p$-adic numbers.
- $\mathbb{Z}_p$: the ring of $p$-adic integers.
- $m$ or $dx$: the Haar measure on $\mathbb{Q}_p$ such that $m(\mathbb{Z}_p) = 1$.
- Any $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{n=\nu_p(x)} a_n p^n \quad (\nu_p(x) \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \text{ and } a_{\nu_p(x)} \neq 0).$$

The fractional part $\{x\}$ of $x$ is defined to be $\sum_{n=\nu_p(x)}^{-1} a_n p^n$.

- Pontryagin duality theorem: the dual group $\hat{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is isomorphic to $\mathbb{Q}_p$. 
Dual group of $\mathbb{Q}_p$

- Fix the following character $\chi \in \widehat{\mathbb{Q}}_p$:

$$\chi(x) = e^{2\pi i \{x\}}.$$  

Notice: $\chi$ is a local constant function ($\chi(x) = 1$, if $x \in \mathbb{Z}_p$).

- For any $y \in \mathbb{Q}_p$, we define

$$\chi_y(x) = \chi(yx).$$

The map $y \mapsto \chi_y$ from $\mathbb{Q}_p$ onto $\widehat{\mathbb{Q}}_p$ is an isomorphism.
Fourier transform

For \( f \in L^1(\mathbb{Q}_p) \), the Fourier transform of \( f \) is defined to be

\[
\hat{f}(y) = \int_{\mathbb{Q}_p} f(x) \overline{\chi_y}(x) \, dx.
\]

Example

\[
\hat{1_{B(0,p^\gamma)}}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi)
\]

\[
\hat{\bigcup_{j=1}^\infty B(c_j, p^\gamma)}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi) \sum_j \chi(-c_j \xi).
\]

Lemma (A criterion of spectral set)

A Borel set \( \Omega \) of finite haar measure is a spectral set with \( \Lambda \) as a spectrum iff

\[
\forall \xi \in \widehat{\mathbb{Q}_p}, \sum_{\lambda \in \Lambda} |\hat{1_{\Omega}}(\lambda - \xi)|^2 = m(\Omega)^2.
\]
Tree structure of $\mathbb{Q}_p$

Vertices $\mathcal{T}$: balls in $\mathbb{Q}_p$.

Edges $\mathcal{E}$: pairs $(B', B) \in \mathcal{T} \times \mathcal{T}$ such that $B' \subset B$, $m(B) = pm(B')$, denote by $B' \prec B$. 

\[ \mathbb{Z}_p \]
\[ p\mathbb{Z}_p \]
\[ p^2\mathbb{Z}_p \]
\[ (p - 1)p + p^2\mathbb{Z}_p \]
\[ (p - 1)+ p\mathbb{Z}_p \]
\[ (p - 1)+ p^2\mathbb{Z}_p \]
\[ p^2 - 1 + p^2\mathbb{Z}_p \]
\[ \frac{1}{p}\mathbb{Z}_p \]

Bounded open sets in $\mathbb{Q}_p$

Any bounded open set $O$ of $\mathbb{Q}_p$ can be described by a subtree $(T_O, E_O)$ of $(T, E)$.

- Let $B^*$ be the smallest ball containing $O$, which will be the root of the tree. For any given ball $B$ contained in $O$, there is a unique sequence of balls $B_0, B_1, \cdots, B_r$ such that

  $$B = B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_r = B^*.$$  

- The set of vertices $T_O$ is composed of all such balls $B_0, B_1, \cdots, B_r$ for all possible balls $B$ contained in $O$.

- The set of edges $E_O$ is composed of all edges $B_i \prec B_{i+1}$ as above.
Any compact open set can be described by a finite tree, because a compact open set is a disjoint finite union of balls of same size. In this case, as in the above construction of subtree we only consider these balls of same size as $B$.

Figure: $\Omega = 3\mathbb{Z}_3 \sqcup (2 + 3\mathbb{Z}_3) \sqcup (4 + 27\mathbb{Z}_3) \sqcup (22 + 27\mathbb{Z}_3)$. 
p-homogenous subsets in \( \mathbb{Q}_p \)

- A subtree \((T', E')\) is said to be homogeneous if the number of descendants of \(B \in T'\) depends only on \(|B|\). If this number is either 1 or \(p\), we call \((T', E')\) a \(p\)-homogeneous tree.
- An bounded open set is said to be homogeneous (resp. \(p\)-homogeneous) if the corresponding tree is homogeneous (resp. \(p\)-homogeneous).
- A bounded open \(p\)-homogenous set must be compact.
p-homogenous subsets in $\mathbb{Q}_p$

Figure: A 2-homogenous tree
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Geometry of Compact Open Spectral Set

Spectra and Translates

**Fuglede Conjecture on $\mathbb{Q}_p$**
Tree structure of $\mathbb{Z}/p^\gamma \mathbb{Z}$

We identify $\mathbb{Z}/p^\gamma \mathbb{Z} = \{0, 1, \cdots, p^\gamma - 1\}$ with $\{0, 1, 2, \cdots p - 1\}^\gamma$ which is considered as a finite tree, denoted by $\mathcal{T}(\gamma)$.

- **Vertices** $\mathcal{T}(\gamma)$: consists of the disjoint union of the sets $\mathbb{Z}/p^n\mathbb{Z}, 0 \leq n \leq \gamma$. Each vertex, except the root of the tree, is identified with a sequence $t_0t_1\cdots t_n$ with $0 \leq n \leq \gamma$ and $t_i \in \{0, 1, \cdots, p - 1\}$.

- **Edges**: consists of pairs $(x, y) \in \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n+1}\mathbb{Z}$ with $x \equiv y \mod p^n$, where $0 \leq n \leq \gamma - 1$.

For example, each point $t$ of $\mathbb{Z}/p^\gamma \mathbb{Z}$ is identified with $t_0t_1\cdots t_{\gamma-1}$, which is a boundary point of the tree.
Tree structure of $\mathbb{Z}/p^\gamma \mathbb{Z}$

Figure: The set $\mathbb{Z}/3^4 \mathbb{Z} = \{0, 1, 2, \cdots, 80\}$ is considered as a tree $\mathcal{T}^{(4)}$. 
$p$-homogenous subsets $\mathbb{Z}/p^\gamma \mathbb{Z}$

Each subset $C \subset \mathbb{Z}/p^\gamma \mathbb{Z}$ will determine a subtree of $T^{(\gamma)}$, denoted by $T_C$, which consists of the paths from the root to the points in $C$. For each $0 \leq n \leq \gamma$, denote

$$C_{\text{mod } p^n} := \{x \in \mathbb{Z}/p^n\mathbb{Z} : \exists y \in C, \text{ such that } x \equiv y \mod p^n\}.$$ 

- **Vertices** $T_C$: consists of the disjoint union of the sets $C_{\text{mod } p^n}$, $0 \leq n \leq \gamma$.

- **Edges**: consists of pairs $(x, y) \in C_{\text{mod } p^n} \times C_{\text{mod } p^{n+1}}$ with $x \equiv y \mod p^n$, where $0 \leq n \leq \gamma - 1$.

The set $C$ is called a $p$-homogenous subsets of $\mathbb{Z}/p^\gamma \mathbb{Z}$ iff the corresponding tree $T_C$ is $p$-homogenous.
$p$-homogenous subsets $\mathbb{Z}/p^\gamma\mathbb{Z}$

**Figure:** For $p = 3, \gamma = 2$, the tree $p$-homogeneous tree determined by \{0, 4, 8, 9, 13, 17, 18, 22, 26\}.
Spectral sets and tiles in $\mathbb{Z}/p^\gamma \mathbb{Z}$

Recall that the Fourier transform of a function $f$ defined on $\mathbb{Z}/p^\gamma \mathbb{Z}$ is defined as

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}/p^\gamma \mathbb{Z}} f(x) e^{-\frac{2\pi i k x}{p^\gamma}}, (\forall k \in \mathbb{Z}/p^\gamma \mathbb{Z}).$$

**Theorem (Fan-F-Shi)** Let $C \in \mathbb{Z}/p^\gamma \mathbb{Z}$. The following statements are equivalent.

1. $C$ is $p$-homogenous.
2. For any $1 \leq i \leq \gamma$, $\#(C_{\text{mod } p^i}) = p^{k_i}$, for some $k_i \in \mathbb{N}$.
3. There exists a subset $I \subset \mathbb{N}$ such that $\#I = \log_p(\#C)$ and $\hat{1}_C(p^\ell) = 0$ for $\ell \in I$.
4. $C$ tiles $\mathbb{Z}/p^\gamma \mathbb{Z}$.
5. $C$ is a spectral set in $\mathbb{Z}/p^\gamma \mathbb{Z}$. 
Spectral sets and tiles in $\mathbb{Z}/p^\gamma\mathbb{Z}$

Figure: For $p = 3$, tree $\mathcal{T}_{l,J}$ with $\gamma = 5$, $l = \{0, 2, 4\}$, $J = \{1, 3\}$.
Compact open spectral sets $\mathbb{Q}_p$

W. l. o. g, we assume that $\Omega$ is of the form

$$\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \cdots, p^\gamma - 1\}$.

Theorem (Fan-F-Shi) The following are equivalent.

1. $\mathcal{T}_C$ is a $p$-homogenous tree.
2. $\Omega$ is $p$-homogenous.
3. $\Omega$ tiles $\mathbb{Q}_p$.
4. $\Omega$ is a spectral set in $\mathbb{Q}_p$. 
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For a subset $\Omega \subset \mathbb{Q}_p$, the set of admissible $p$-orders of $\Omega$:

$$l_{\Omega} := \{ i \in \mathbb{Z}, \exists x, y \in \Omega \text{ such that } |x - y|_p = p^{-i} \}.$$

Remark

- For compact open $\Omega$, $\exists \gamma \in \mathbb{Z}$ such that $i \in l_{\Omega}$ if $i \geq \gamma$.
- For $p$-homogenous compact open set $\Omega$, an integer $i \in l_{\Omega}$ iff the balls of radius $p^{-i}$ in the tree $T_{\Omega}$ has $p$ descendants.
- For two $p$-homogenous compact open set $\Omega$ and $\Omega'$, $l_{\Omega} = l_{\Omega'}$ iff $\exists$ isometric transformation $f : \mathbb{Q}_p \to \mathbb{Q}_p$ such that $f(\Omega) = \Omega'$. 
Spectra and Translates

For a discrete subset $E$ in $\mathbb{Q}_p$, we call $E$ a uniformly discrete set if $l_E$ is upper bounded. For each integer $n$, denote

$$l^n_E := \{ i \in l_E : i \geq -n \}.$$

A uniformly discrete set $E$ is called $p$-homogenous discrete if

$$\#(E \cap B(a, p^n)) = p^{\#l^n_E} \text{ or } 0, \quad \forall a \in \mathbb{Q}_p.$$

Remark
Let $E$ and $E'$ be two $p$-homogenous discrete subset in $\mathbb{Q}_p$. Then $l_E = l_{E'}$ iff $\exists$ isometric transformation $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ such that $f(E) = E'$. 
Spectra and Translates

Theorem (Fan-F-Shi)

Let $\Omega$ be a $p$-homogenous compact open set. Assume that $\Omega$ is a spectral pair and $(\Omega, T)$ is a tiling pair.

- Each spectrum $\Lambda$ is $p$-homogenous discrete with $I_{\Lambda} = -(I_\Omega + 1)$.
- Each translate $T$ is $p$-homogenous discrete with $I_T = \mathbb{Z} \setminus I_\Omega$. 

Spectra and Translates

Theorem (Fan-F-Shi)

Let $\Omega$ be a $p$-homogeneous compact open set in $\mathbb{Q}_p$.

- Subject to an isometric bijection,

$$\Lambda = \sum_{i \in I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^{-i-1} \subset \mathbb{Q}_p$$

is the unique spectrum of $\Omega$.

- Subject to an isometric bijection,

$$T = \sum_{i \notin I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^i \subset \mathbb{Q}_p$$

is the unique tiling complement of $\Omega$. 
Spectra and Translates

\[ \Omega = 2 + 4\mathbb{Z}_2 \sqcup 3 + 4\mathbb{Z}_2 \]

**Figure:** Consider the translate \( T \) of \( \Omega \) as an infinite tree and the points of \( T \) are the boundary points of the tree.
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Fuglede Conjecture on \( \mathbb{Q}_p \)

A set \( \Omega \subset \mathbb{Q}_p \) is called an almost compact open set, if \( \exists \) compact open \( \Omega' \subset \mathbb{Q}_p \) such that

\[
\text{m}(\Omega \setminus \Omega') = \text{m}(\Omega' \setminus \Omega) = 0.
\]

**Theorem**

A Borel set \( \Omega \in \mathbb{Q}_p \) is a spectral set if and only if it tiles \( \mathbb{Q}_p \). Moreover, \( \Omega \) is an almost compact open set.

**Theorem**

A subset \( E \) of \( \mathbb{Q}_p \) is a spectrum iff it is a translate. Moreover, it is a \( p \)-homogenous discrete subset of \( \mathbb{Q}_p \).
Idea of Proof

- Calculate the densities of spectrum $\Lambda$ and translate $T$.
- Consider $\mu_\Lambda$ and $\mu_T$ as distributions in $\mathbb{Q}_p$ (continuous linear functionals on space of the local constant function with compact support). Fourier analysis on $\mathbb{Q}_p$ (See Ableverio Khrennikov and Shelkovich’ s book 2010, Vladimirov and Volovich and Zelenov’ s book 1994).
- Analyze the zeros of $\hat{\mu}_\Lambda$ and $\hat{\mu}_T$. Obtain the structure of $\Lambda$ and $T$ ($p$-homogenous discrete).
The end

Thank you!