Spectral Problem in Local Fields

Ai-Hua FAN

University of Picardie, France/CCNU, China

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Outline

Introduction

Spectral problem on l.c.a. group

Notation

1. G : locally compact Abelian group ; $\widehat{G}$: dual group of G
2. $\Omega \subset G$: Borel set, $0 < \mathfrak{m}(\Omega) < \infty$ ($\mathfrak{m}$: Haar measure).
3. $L^2(\Omega)$: Hilbert space of square Haar-integrable functions on Ω.

Definition Ω is a spectral set if $\exists \Lambda \subset \widehat{G}$ countable which form a Hilbert basis of $L^2(\Omega)$. Λ is called a spectrum of Ω. (Ω, Λ) is a spectral pair.

Definition Ω tiles G by translation if $\exists T \subset G$ countable such that

$$\sum_{t \in T} 1_\Omega(x - t) = 1 \quad \mathfrak{m}-a.e.\ x \in G \quad i.e. \quad 1_\Omega * 1_T = 1.$$

Such a set T is called a tiling complement of Ω.

Spectral set conjecture : Ω **is a spectral set if and only if** Ω **tiles** G **by translation.**

Remarks

1. Tiling problem in finite group : $A \oplus B = G$.
2. Spectral problem in finite group : $\left(\frac{1}{\sqrt{\Omega}}\chi_\lambda(\omega)\right)_{\Lambda\times\Omega}$ is Hadamard matrix.
3. Three questions : Spectral sets $=?$, Tiles $=?$, $\{\text{spectral} \quad \text{sets}\} = \{\text{tiles}\}$?
4. For general finite group, there is no complete solutions.
5. $\mathbb{Z}/m\mathbb{Z}$?
6. $\mathbb{Z}/2\cdot 3\cdot 5\mathbb{Z}$: there are 2^{30} subsets. But $|A|\cdot|B| = |G|$ so $|A|$ divides $|G|$.
7. $\mathbb{Z}/p^n\mathbb{Z}$: the conjecture is true and we know the tiles ($=$ spectral sets).
8. $\mathbb{R}^d$ $(d \geq 3)$: the conjecture is not true.
9. $\mathbb{R}^d$ $(d = 1, 2)$: open problem.
10. $\mathbb{Q}_p$: our aim is to prove that the conjecture is true ($\rightarrow$ Shilei FAN).

Spectral measures on l.c.a. group

Notation

1. μ : Borel probability measure on G
2. $L^2(\mu)$: Hilbert space of μ-square integrable functions on G.

Definition μ is a spectral measure if $\exists \Lambda \subset \widehat{G}$ countable which form a Hilbert basis of $L^2(\mu)$. Λ is called a spectrum of μ. (μ, Λ) is a spectral pair.

Spectral problem : **which measure are spectral measures ?**.

NB Let $\mu_\Omega = \frac{\mathfrak{m}|_\Omega}{\mathfrak{m}(\Omega)}$. Then μ_Ω is spectral iff Ω is spectral.

Criterion on space K^d

Notation

1. K : locally compact fields ($\mathbb{R}$, $\mathbb{Q}_p$, local fields)
2. G : $G = K^d$; $\widehat{K^d} \cong K^d$. For each character χ_ξ,

$$\widehat{\mu}(\xi) = \int \overline{\chi_\xi(x)} d\mu(x).$$

Theorem (Jorgensen-Petersen 1998) : (μ, Λ) is a spectral pair if and only if

$$\forall \xi \in K^d, \quad \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi - \lambda)|^2 = 1 \quad i.e. \quad |\widehat{\mu}|^2 * 1_\Lambda = 1.$$

NB : Similar criterion holds on finite abelian groups.
NB : Convolution equation : Given $f \in L^1$ and $g \in L^1_{\text{loc}}$. Find a Radon measure μ (or a distribution) such that

$$f * \mu = g.$$

Examples of spectral sets in $\mathbb{R}^d$

1. (Trivial) $\Omega = [0,1] \subset \mathbb{R}$. $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2(\Omega)$.And

$$[0,1] \oplus \mathbb{Z} = \mathbb{R}.$$

2. (Non-trivial) $\Omega = [0,1] \cup [2,3] \subset \mathbb{R}$. $\{e^{2\pi i\lambda t}\}_{\lambda\in\mathbb{Z}\cup(\mathbb{Z}+1/4)}$ is a Hilbert basis of $L^2(\Omega)$. And

$$[0,1] \cup [2,3] \oplus 4\mathbb{Z} \cup \{1\} = \mathbb{R}.$$

Question. **Are there other sets Ω in $\mathbb{R}^d$ such that $L^2(\Omega)$ has a basis of exponential functions $\{e^{2\pi i\lambda\cdot t}\}_{\lambda\in\Lambda}$?**

1. For $[a,b] \times [c,d]$, we can take $\Lambda = \frac{1}{b-a}\mathbb{Z} \times \frac{1}{d-c}\mathbb{Z}$.
2. $[0,1] \cup [2,4]$ in $\mathbb{R}$ is not
3. Triangles and disks in $\mathbb{R}^2$ are not.

Finite Abelian groups $\mathbb{Z}/p^\gamma\mathbb{Z}$

Assume that $p \geq 2$ is a prime.

Theorem (Fan-Fan-Shi 2015) : Let

$$\Omega \subset \{0, 1, 2, \cdots, p^\gamma - 1\} \cong \mathbb{Z}/p^\gamma\mathbb{Z}.$$

Then Ω is a spectral set if and only if it tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$. Such a tile Ω is characterized by

$$\forall 1 \leq n \leq \gamma, \quad \sharp(C \mod p^n) \text{ is a power of p.}$$

Examples $C = \{0, 3, 4, 7\} \subset \mathbb{Z}/2^3\mathbb{Z}$ is a spectral set (i.e. a tile) for

$$C \mod 2 = \{0, 1\}, \quad C \mod 2^2 = \{0, 3\}, \quad C \mod 2^3 = C.$$

But $D = \{0, 3, 4, 5\}$ is not for $\sharp(D \bmod 2^2) = 3$.

NB **The problem is not solved for general $\mathbb{Z}/m\mathbb{Z}$.**

Fuglede Conjecture on $\mathbb{R}^d$

Fuglede Conjecture on $\mathbb{R}^d$

Assume $\Omega \subset \mathbb{R}^d$ and $0 < |\Omega| < \infty$ ($|\cdot|$ denoting the volume).

Fuglede conjecture : **Ω is a spectral set in $\mathbb{R}^d$ if and only if Ω tiles $\mathbb{R}^d$ by translation**.

Fuglede Theorem (1974) : Let $\Lambda := A\mathbb{Z}^d$ ($A \in GL(\mathbb{R}, d)$) be a lattice in $\mathbb{R}^d$. (Ω, Λ) is a spectral pair if and only if Ω tiles $\mathbb{R}^d$ by the dual lattice

$$\Lambda^* := \{\lambda' \in \mathbb{R}^d : \forall \lambda \in \Lambda, \ \langle \lambda', \lambda \rangle \in \mathbb{Z}\}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

State of the art (T. Tao, M. Matolcsi, Kolountzakis-M. Matolcsi) **The conjecture is false when $d \geq 3$. It is open when $d = 1, 2$.**

Segal problem

Segal Problem(1958) Let $\Omega \subset \mathbb{R}^d$ be a bounded open connected. On which Ω there exist, on the Hilbert space $L^2(\Omega)$, commuting self-adjoint restrictions of the operators $-i\partial_1, \cdots, -i\partial_d$ (acting on $L^2(\Omega)$ in the distribution sense) ?

On such a region Ω, constant-coefficient linear partial differential operators are normal operators in $L^2(\Omega)$.

Theorem of Fuglede (1974) Suppose $\Omega \subset \mathbb{R}^d$ be a Nikodym region (bounded open connected on which every distribution of finite Dirichlet integral is itself in $L^2(\Omega)$). The problem of Segal has an affirmative answer if and only if $L^2(\Omega)$ has a Hilbert space consisting of exponential functions.

$\rightarrow$ **Fuglede conjecture (1974)** Ω's are tilings by translations.

Results on Fuglede Conjecture

1. Lagarias-Wang (1996) : A large of tiles in $\mathbb{R}$ are spectral ; A Tijdeman conjecture implies all tiles in $\mathbb{R}$ are spectral.
2. Kolountzaki (1999) : Non-symmetric convex sets are not spectral.
3. Laba (2001) : Fuglede conjecture holds for union of two intervals.
4. Bose-Kumar-Krishnan-Madan (2010,2011) : Union of three intervals.
5. Iosevich-Katz-Tao (2003) : Fuglede conjecture holds for convex sets in $\mathbb{R}^2$.
6. Many works on (singular) spectral measures : Jorgensen, Pedersen, Strichartz, Wang, Lau, Dutkay et al.
7. Hilbert basis, Riesz basis, Fourier frame etc.

Spectral problem on local fields

Results on Fuglede Conjecture

Notation

1. K : local field of absolute value $|\cdot|$
2. $\mathfrak{D}$: ring of integers in K, i.e. unit ball $\{x \in K : |x| \leq 1\}$.
3. $\mathfrak{m}$ or dx : Haar measure, $\mathfrak{m}(\mathfrak{D}) = 1$.
4. $\mathfrak{P} = (\mathfrak{p}) = \mathfrak{p}\mathfrak{D}$: ball $\{x \in K : |x| < 1\}$, i.e. the maximal ideal in $\mathfrak{D}$.
5. $\mathbf{k} \cong \mathbb{F}_q$: residue class field $\mathfrak{D}/\mathfrak{P}$
6. $\mathfrak{p}$: a prime element of K
7. $\widehat{K} \cong K$: dual group of $(K, +)$
8. $\chi \in \widehat{K}$: equal to 1 on $\mathfrak{D}$, non-constant on $\mathfrak{p}^{-1}\mathfrak{D}$.
9. $\chi_y(x) = \chi(yx)$: $y \mapsto \chi_y$ from K onto $\widehat{K}$ is an isomorphism.
10. K^d : is endowed with the norm $|x| = \max_{1 \leq j \leq d} |x_j|$.
11. $x \cdot y = x_1 y_1 + \cdots + x_d y_d$.

Relaxation of orthogonality

Definition $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is a Fourier frame of $L^2(\mu)$ if there exist constants $A > 0$ and $B > 0$ such that

$$A\|f\|^2 \le \sum_\lambda |\langle f, \chi_\lambda\rangle_\mu|^2 \le B\|f\|^2, \qquad \forall f \in L^2(\mu) \tag{1}$$

Λ is called a F-spectrum.

Definition If only the first inequality (resp. the second inequality) in satisfied in (1), we say that Λ is a set of sampling (resp. a Bessel sequence) of $L^2(\mu)$.

Definition If for any sequence $\{a_\lambda\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)$, there exists $f \in L^2(\Omega)$ such that $a_\lambda = \widehat{f}(\lambda)$ for all $\lambda \in \Lambda$, we say that Λ is a set of interpolation for $L^2(\Omega)$.

Remark $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is orthogonal in $L^2(\mu)$ iff

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}_\mu := \{\xi \in \widehat{K}^d : \widehat{\mu}(\xi) = 0\}. \tag{2}$$

Because

$$\langle \chi_\xi, \chi_\lambda\rangle_\mu = \int \chi_\xi \overline{\chi}_\lambda d\mu = \widehat{\mu}(\lambda - \xi).$$

Purity of F-spectral measures

Theorem A compactly supported F-spectral measure on K^d must be of one of the following types :

finitely discrete,

singularly continuous,

absolutely continuous.

Theorem Let μ be a compactly supported and absolutely continuous probability measure on K^d. If μ is a F-spectral measure, then

$$A \leq \frac{d\mu}{dx} \leq B$$

almost everywhere on the support of μ, where $0 < A \leq B < \infty$ are two constants.

NB Proofs are based on the perturbation of Bessel sets and the spectrum of Landau operators.

Perturbation of Bessel sequences

Recall that Λ is a Bessel sequence of $L^2(\mu)$ if there exists a constant $B > 0$ such that

$$|\langle f, \chi_\lambda \rangle_\mu|^2 \leq B\|f\|^2, \qquad \forall f \in L^2(\mu) \tag{3}$$

Theorem Let $\{\lambda_n\}$ be a Bessel sequence of $L^2(\mu)$ where μ is of compact support. Let $\{\gamma_n\}$ be another sequence. Suppose there exists a constant $C > 0$ such that

$$\forall n, \quad |\gamma_n - \lambda_n| \leq C.$$

Then $\{\gamma_n\}$ is also a Bessel sequence of $L^2(\mu)$.

Proof

Lemma 1. If $\{\lambda_n\}$ is a Bessel sequence of $L^2(\mu)$, then$\{a^{-1}\lambda_n\}$ $(a \in K^*)$ is a Bessel sequence of $L^2(\mu \circ \tau^{-1})$ $(\tau x = ax)$.

Lemma 2. Suppose μ has its support in $\mathcal{D}^d$. If $\{\lambda_n\}$ is a Bessel sequence of $L^2(\mu)$, so is $\{\lambda_n + \eta_n\}$ for any sequence $\{\eta_n\}$ such that $|\eta_n| \leq 1$.

Lemma 3. Assume $\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_r$. Then Λ is a Bessel sequence of $L^2(\mu)$ iff Λ_j $(j = 1, 2, \cdots, r)$ are Bessel sequences of $L^2(\mu)$.

Proof. Assume $\mathrm{suppt}\mu \subset \mathcal{D}^d$ (Lemma 1). Consider the difference $\delta_n = \gamma_n - \lambda_n$. Since $|\delta_n| \leq C$, we can write

$$\delta_n = t_n + \eta_n, \qquad (t_n \in D; \eta_n \in \mathcal{D}^d)$$

where D is a finite set of representatives of $K^d/\mathcal{D}^d$. Then

$$\gamma_n = (\lambda_n + t_n) + \eta_n$$

Then we partition the sequences according to $t_n = d$ with $d \in D$. We conclude by Lemma 2 and Lemma 3.

Beurling density of spectrum

Definition The upper/lower Beurling density of a discrete set Γ in K^d :

$$D^+(\Gamma) = \limsup_{n\to\infty} \sup_{x\in K^d} \frac{\sharp(\Gamma\cap B(x,q^n))}{q^{dn}},$$
$$D^-(\Gamma) = \liminf_{n\to\infty} \inf_{x\in K^d} \frac{\sharp(\gamma\cap B(x,q^n))}{q^{dn}}.$$

we can replace $B(x,q^n)$ by $x+\mathfrak{p}^{-n}I$ where I is any compact set such that $\mathfrak{m}(I)=1$. The values $D^+(\Gamma)$ and $D^-(\Gamma)$ don't depend on I. If $D^+(\Gamma)=D^-(\Gamma)$, the common value is the Beurling density.

Theorem Let $\Omega\subset K^d$, $0<\mathfrak{m}(\Omega)<\infty$. and Let $\Lambda\subset\widehat{K}^d$ be discrete.
(1) If Λ is a set of sampling of $L^2(\Omega)$, then $D^-(\Lambda)\geq\mathfrak{m}(\Omega)$.
(2) If Λ is a set of interpolation of $L^2(\Omega)$, then $D^+(\Lambda)\leq\mathfrak{m}(\Omega)$.
(3) If Λ is a F-spectrum of Ω, then $D(\Lambda)=\mathfrak{m}(\Omega)$.

H. Landau operators

1. $\Omega \subset K^d$, $\Delta \subset \widehat{K}^d : 0 < \mathfrak{m}(\Omega) < \infty, \quad 0 < \mathfrak{m}(\Delta) < \infty.$
2. $L^2(\Omega) \subset L^2(K^d)$.
3. $L^2(\Omega)\widehat{\ } := \{g \in L^2(\widehat{K}^d) : \exists f \in L^2(\Omega) \text{ such that } g = \widehat{f}\}.$
4. $\check{f}(x)$: the inverse Fourier transform of $f \in L^1(K^d)$.

Definition Define $T_\Omega : L^2(\widehat{K}^d) \to L^2(\Omega)\widehat{\ } \subset L^2(\widehat{K}^d)$:

$$T_\Omega g = (1_\Omega \check{g})\widehat{\ }. \quad (\text{restriction on } \Omega, \text{space side })$$

Let P_Δ be the orthogonal projection from $L^2(\widehat{K}^d)$ onto $L^2(\Delta) \subset L^2(\widehat{K}^d)$:

$$P_\Omega g(\xi) = 1_\Delta(\xi) g(\xi). \quad (\text{restriction on } \Delta, \text{Fourier side })$$

The Landau operator $\mathcal{L} = \mathcal{L}_{\Omega,\Delta} : L^2(\widehat{K}^d) \to L^2(\widehat{K}^d)$ is defined by

$$\mathcal{L} = T_\Omega P_\Delta T_\Omega.$$

The auxiliary Landau operator $\mathcal{L}^\sharp : L^2(\widehat{K}^d) \to L^2(\widehat{K}^d)$ is defined by

$$\mathcal{L}^\sharp = P_\Delta T_\Omega P_\Delta.$$

Properties of Landau operators

Proposition $\mathcal{L}$ is a positive compact self-adjoint operator, a Hilbert-Schmidt integral operator with kernel

$$K(\eta,\xi) = \int 1_\Delta(t)\widehat{1_\Omega}(\eta - t)\overline{\widehat{1_\Omega}(\xi - t)}dt.$$

Eigenvalues of $\mathcal{L}$ and of $\mathcal{L}^\sharp$

$$1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \cdots \to 0$$

Courant-Fischer-Weyl min-max principle

$$\lambda_k = \max_{S_k} \min_{x\in S_k} \frac{\|\mathcal{L}x\|}{\|x\|} = \min_{S_{k-1}} \max_{x\in S_{k-1}^\perp} \frac{\|\mathcal{L}x\|}{\|x\|} \quad (\dim S_k = k).$$

Eigenvalues $\Lambda_k(\Omega, \Delta)$

1. $\lambda_k(\Omega, \Delta) = \lambda_k(\Omega + \sigma, \Delta + \tau)$ for all $\sigma, \tau \in K^d$.
2. $\lambda_k(\Omega, \Delta) = \lambda_k(a\Omega, a^{-1}\Delta)$ for any $a \in K^*$.
3. $\lambda_k(\Omega, \Delta) = \lambda_k(\Delta, \Omega)$.
4. $\lambda_k(\Omega, \Delta_1) \leq \lambda_k(\Omega, \Delta_2)$ if $\Delta_1 \subset \Delta_2$.
5. $\sum_{k=1}^{\infty} \lambda_k(\Omega, \Delta) = \mathfrak{m}(\Omega)\mathfrak{m}(\Delta)$.
6. $\sum_{k=1}^{\infty} \lambda_k(\Omega, \Delta)^2 = \int_\Delta \int_\Delta |\widehat{1_\Omega}(u - v)|^2 du dv$.
7. For $\Delta = \Delta_1 \sqcup \Delta_2$, we have
$$\sum_{k=1}^{\infty} \lambda_k(\Omega, \Delta)^2 \geq \sum_{k=1}^{\infty} \lambda_k(\Omega, \Delta_1)^2 + \sum_{k=1}^{\infty} \lambda_k(\Omega, \Delta_2)^2.$$
8. $\lambda_{k+1}(\Omega, \Delta) \leq \sup \left\{ \|P_\Delta f\|^2 : \|f\| = 1, f \in L^2(\Omega)\hat{} \cap C_k^\perp \right\}$,
9. $\lambda_k(\Omega, \Delta) \geq \inf \left\{ \|P_\Delta f\|^2 : \|f\| = 1, f \in C_k \right\}$
(C_k is any k-dimensional subspace of $L^2(K^d)$).

$\Lambda_k(\Omega, \Delta)$ when $\Delta \in \mathcal{A}_a, \Omega \in \mathcal{A}_b$

$\mathcal{A}_a$: family of finite union of ball of radius q^a.
Lemma 1 Assume $\Omega = B(0, q^b)$ and $\Delta = B(0, q^a)$ with $a + b \geq 0$.

1. 1 and 0 are the only eigenvalues.
2. Multiplicity of 1 : q^{a+b}.
3. Eigenvectors of 1 : 1_B's (B balls in $B(0, q^a)$ of radius q^{-b}).

It follows that

$$\mathcal{L}f(x) = \sum_{c \in \mathfrak{p}^b \mathbb{L}_{a+b}} 1_{B(c,q^b)}(x) \cdot q^{-db} \int_{B(c,q^b)} f(y)dy.$$

Lemma 2 Let $\Delta \in \mathcal{A}_a$ and $\Omega \in \mathcal{A}_b$ with $a + b \geq 0$. Then

$$\lambda_k(\Omega, \Delta) = \begin{cases} 1 & \text{if} \quad 1 \leq k \leq \mathfrak{m}(\Omega)\mathfrak{m}(\Delta), \\ 0 & \text{if} \quad \qquad k > \mathfrak{m}(\Omega)\mathfrak{m}(\Delta). \end{cases}$$

Fuglede conjecture holds in $\mathbb{Q}_p$

Theorem (Fan-Fan-Shi 2015) Let Ω be a Borel set in $\mathbb{Q}_p$ such that $0 < m(\Omega) < \infty$. Then

1. Ω is a spectral set if and only if it tiles $\mathbb{Q}_p$.
2. In this case, Ω is an open compact set, up to a set of zero Haar measure.
3. Such spectral sets are characterized by a simple homogeneous tree structure.

NB The Fuglede conjecture is false in $\mathbb{Q}_p^d$ when $d > d_p$. But we don't know the critical value d_p.