Order, type and cotype of growth for p-adic entire functions

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We denote by $\mathbb{K}$ an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $| \, . \, |$. Analytic functions inside a disk or in the whole field $\mathbb{K}$ were introduced and studied in many books. Given $\alpha \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in \mathbb{K} \mid |x-\alpha| \leq R\}$, by $d(\alpha, R^-)$ the disk $\{x \in \mathbb{K} \mid |x - \alpha| < R\}$, by $C(\alpha, r)$ the circle $\{x \in \mathbb{K} \mid |x - \alpha| = r\}$, by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ (i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$). Given $f \in \mathcal{M}(\mathbb{K})$, we will denote by $q(f, r)$ the number of zeros of f in $d(0, r)$, taking multiplicity into account and by $u(f, r)$ the number of distinct multiple zeros of f in $d(0, r)$. Throughout the paper, log denotes the Neperian logarithm.

Here we mean to introduce and study the notion of order of growth and type of growth for functions of order t. We will also introduce a new notion of cotype of growth in relation with the distribution of zeros in disks which plays a major role in processes that are quite different from those in complex analysis. This has an application to the question whether an entire function can be devided by its derivative inside the algebra of entire functions.

Similarly to the definition known on complex entire functions, given $f \in \mathcal{A}(\mathbb{K})$, the superior limit

$$\limsup_{r \to +\infty} \frac{\log(\log(|f|(r)))}{\log(r)}$$

is called *the order of growth of f* or *the order of f* in brief and is denoted by $\rho(f)$. We say that f has *finite order* if $\rho(f) < +\infty$.

Theorem 1: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then:*
$\rho(f+g) \leq \max(\rho(f), \rho(g))$,
$\rho(fg) = \max(\rho(f), \rho(g))$,

Corollary 1.1: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then $\rho(f^n) = \rho(f)\ \forall n \in \mathbb{N}^*$. If $\rho(f) > \rho(g)$, then $\rho(f+g) = \rho(f)$.*

Remark: ρ is an ultrametric extended semi-norm.

Notation: Given $t \in [0, +\infty[$, we denote by $\mathcal{A}(\mathbb{K}, t)$ the set of $f \in \mathcal{A}(\mathbb{K})$ such that $\rho(f) \leq t$ and we set

$$\mathcal{A}^0(\mathbb{K}) = \bigcup_{t \in [0, +\infty[} \mathcal{A}(\mathbb{K}, t).$$

Corollary 1.2. *For any $t \geq 0$, $\mathcal{A}(\mathbb{K}, t)$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(\mathbb{K})$. If $t \leq u$, then $\mathcal{A}(\mathbb{K}, t) \subset \mathcal{A}(\mathbb{K}, u)$ and $\mathcal{A}^0(\mathbb{K})$ is also a $\mathbb{K}$-subalgebra of $\mathcal{A}(\mathbb{K})$.*

Theorem 2 *Let $f \in \mathcal{A}(\mathbb{K})$ and let $P \in \mathbb{K}[x]$. Then $\rho(P \circ f) = \rho(f)$ and $\rho(f \circ P) = \deg(P)\rho(f)$.*

Theorem 3: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be transcendental. If $\rho(f) \neq 0$, then $\rho(f \circ g) = +\infty$. If $\rho(f) = 0$, then $\rho(f \circ g) \geq \rho(g)$.*

Theorem 4 *Let $f \in \mathcal{A}(\mathbb{K})$ be not identically zero. If there exists $s \geq 0$ such that*

$$\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^s} \right) < +\infty$$

then $\rho(f)$ is the lowest bound of the set of $s \in [0, +\infty[$ such that

$$\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^s} \right) = 0.$$

Moreover, if

$$\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^t} \right)$$

is a number $b \in]0, +\infty[$, then $\rho(f) = t$. If there exists no s such that

$$\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^s} \right) < +\infty,$$

then $\rho(f) = +\infty$.

Example: Suppose that for each $r > 0$, we have $q(f,r) \in [r^t \log r, r^t \log r + 1]$. Then of course, for every $s > t$, we have

$$\limsup_{r \to +\infty} \frac{q(f,r)}{r^s} = 0$$

and $\limsup_{r\to+\infty} \frac{q(f,r)}{r^t} = +\infty$, so there exists no $t > 0$ such that $\frac{q(f,r)}{r^t}$ have non-zero superior limit $b < +\infty$.

Definition and notation: Let $t \in [0, +\infty[$ and let $f \in \mathcal{A}(\,\mathbb{K})$ of order t. We set

$$\psi(f) = \limsup_{r\to+\infty} \frac{q(f,r)}{r^t}$$

and call $\psi(f)$ the *cotype* of f.

Theorem 5 *Let $f,\ g \in \mathcal{A}^0(\,\mathbb{K})$ be such that $\rho(f) = \rho(g)$. Then*

$$\max(\psi(f), \psi(g)) \le \psi(fg) \le \psi(f) + \psi(g).$$

Theorem 6 is similar to a well known statement in complex analysis and its proof also is similar when $\rho(f) < +\infty$ [10] but is different when $\rho(f) = +\infty$.

Theorem 6 *Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\,\mathbb{K})$. Then*
$\rho(f) = \limsup_{n\to+\infty} \Big(\frac{n\log(n)}{-\log|a_n|}\Big)$.

Remark: Of course, polynomials have a growth order equal to 0. On $\mathbb{K}$ as on $\mathbb{C}$ we can easily construct transcendental entire functions of order 0 or of order ∞.

Example 1: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $-\log|a_n| \in [n(\log n)^2, n(\log n)^2+1]$. Then clearly,

$$\lim_{n\to+\infty} \frac{\log|a_n|}{n} = -\infty$$

hence the function $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence equal to $+\infty$. On the other hand,

$$\lim_{n\to+\infty} \frac{n\log n}{-\log|a_n|} = 0$$

hence $\rho(f) = 0$.

Example 2: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $-\log|a_n| \in [n\sqrt{\log n}, n\sqrt{\log n}+1]$. Then

$$\lim_{n\to+\infty} \frac{\log|a_n|}{n} = -\infty$$

again and hence the function $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence equal to $+\infty$. On the other hand,

$$\lim_{n\to+\infty}\Big(\frac{n\log n}{-\log|a_n|}\Big) = +\infty$$

hence $\rho(f) = +\infty$.

Here, we must recall a theorem proven in 2010 to characterize meromorphic admitting a primitive:

Theorem 7: *Let $f \in \mathcal{M}(\mathbb{K})$. Then f admits primitives if and only if all its residues are null.*

The following theorem was proven in 2011 with help of Jean-Paul Bezivin:

Theorem 8: *Let $f \in \mathcal{M}(\mathbb{K})$. Suppose that there exists $s \in]0, +\infty[$ such that $u(f, r) < r^s \ \forall r > 1$. Then, for every $b \in \mathbb{K}$, $f' - b$ has infinitely many zeros.*

Thanks to Theorem 8, we can prove Theorem 9:

Theorem 9: *Let $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$ with $g \in \mathcal{A}(\mathbb{K})$ and $h \in \mathcal{A}^0(\mathbb{K})$ and $\psi(h) < +\infty$. Then for every $b \in \mathbb{K}$, $f' - b$ has infinitely many zeros.*

Proof: Set $t = \rho(h)$. There exists $\ell > \psi(h)$ such that $q(r,h) \leq \ell r^t \; \forall r > 1$. Consequently, taking $s > t$ big enough, we have $u(f,r) < r^s \; \forall r > 1$ and hence f satisfies the hypotheses of Theorem 8. Therefore, for every $b \in K$, $f' - b$ has infinitely may zeros.

Corollary 9.1: *Let $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$ have all its residues null, with $g \in \mathcal{A}(\mathbb{K})$ and $h \in \mathcal{A}^0(\mathbb{K})$ and $\psi(h) < +\infty$. Then for every $b \in \mathbb{K}$, $f - b$ has infinitely many zeros.*

Remark: Consider a function f of the form $\sum_{n=1}^{\infty} \frac{1}{(x-a_n)^2}$ with $|a_n| = n^t$. Clearly f belongs to $\mathcal{M}(\mathbb{K})$, all residues are null, hence f admits primitives. Next, primitives satisfy the hypothesis of Theorem 8. Consequently, f takes every value infinitely many times. Therefore, f cannot be of the form $\frac{P}{h}$ with $P \in \mathbb{K}[x]$ and $h \in \mathcal{A}(\mathbb{K})$.

Definition and notation: In complex analysis, the type of growth is defined for an entire function of order t as

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log(M_f(r))}{r^t},$$

with $t < +\infty$. Of course the same notion may be defined for $f \in \mathcal{A}(\mathbb{K})$. Given $f \in \mathcal{A}^0(\mathbb{K})$ of order t, we set $\sigma(f) = \limsup\limits_{r\to+\infty} \dfrac{\log(|f|(r))}{r^t}$ and $\sigma(f)$ is called *the type of growth of f*.

Similarly, we set $\widetilde{\sigma}(f) = \liminf\limits_{r\to+\infty} \dfrac{\log(|f|(r))}{r^t}$.

Theorem 10: *Let $f,\ g \in \mathcal{A}^0(\mathbb{K})$. Then $\sigma(fg) \leq \sigma(f) + \sigma(g)$ and $\sigma(f+g) \leq \max(\sigma(f), \sigma(g))$. If $\rho(f) = \rho(g)$, then $\max(\sigma(f), \sigma(g)) \leq \sigma(fg)$ and if $c|f|(r) \geq |g|(r)$ with $c > 0$ when r is big enough, then $\sigma(f) \geq \sigma(g)$.*

Corollary 10.1: *Let $f,\ g \in \mathcal{A}^0(\mathbb{K})$ be such that $\rho(f) = \rho(g)$ and $\sigma(f) > \sigma(g)$. Then $\sigma(f+g) = \sigma(f)$.*

Theorem 11: *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}^0(\mathbb{K})$ *such that* $\rho(f) \in]0, +\infty[$. *Then*

$$\sigma(f)\rho(f)e = \limsup_{n\to+\infty} \left(n \sqrt[n]{|a_n|^t}\right).$$

Notation: Let $f \in \mathcal{A}(\mathbb{K})$, let $(a_n)_{n\in \mathbb{N}}$ be the sequence of zeros of f with $|a_n| \leq |a_{n+1}|$, $n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let w_n be the multiplicity order of a_n. For every $r > 0$, let $k(r)$ be the integer such that $|a_n| \leq r\ \forall n \leq k(r)$ and $|a_n| > r\ \forall n > k(r)$. . We set $\psi(f, r) = \sum_{n=0}^{k(r)} \frac{w_n}{r^t}$ and $\sigma(f, r) = \sum_{n=0}^{k(r)} \frac{w_n(\log(r)-\log(c_n))}{r^t}$.

Theorem 12: *Let* $f \in \mathcal{A}^0(\mathbb{K})$ *be not identically zero. Then*

$$\rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)\Big(e\sigma(f) - \widetilde{\sigma}(f)\Big).$$

Moreover, if $\psi(f) = \lim_{r\to+\infty} \frac{q(f,r)}{r^{\rho(f)}}$ *or if* $\sigma(f) = \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^{\rho(f)}}$, *then* $\psi(f) = \rho(f)\sigma(f)$.

Proof: Without loss of generality we can assume that $f(0) \neq 0$. Let $t = \rho(f)$ and set $\ell = \log(|f(0|)$.

Let $(a_n)_{n\in\,\mathbb{N}}$ be the sequence of zeros of f with $|a_n| \leq |a_{n+1}|$, $n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let w_n be the multiplicity order of a_n. For every $r > 0$, let $k(r)$ be the integer such that $|a_n| \leq r\ \forall n \leq k(r)$ and $|a_n| > r\ \forall n > k(r)$. Then by Theorem A, we have $\log(|f|(r)) = \ell + \sum_{n=0}^{k(r)} w_n(\log(r) - \log(|a_n|))$ hence

$$\sigma(f) = \limsup_{r\to+\infty} \Big(\frac{\ell + \sum_{n=0}^{k(r)} w_n(\log(r) - \log(|a_n|))}{r^t}\Big).$$

Given $r > 0$, set $c_n = |a_n|$, and let us keep the notations above. Then

$$\sigma(f) = \limsup_{r\to+\infty} \sigma(f,r), \quad \psi(f) = \limsup_{r\to+\infty} \psi(f,r).$$

First we will show the inequality $\psi(f) \geq \rho(f)\sigma(f)$. Let us fix $\alpha > 0$. We can write

$$\sigma(f,r) = \sum_{n=0}^{k(\frac{r}{e^\alpha})} \frac{w_n(\log(r) - \log(\frac{r}{e^\alpha}))}{r^t}$$

$$+ \sum_{j=0}^{k(\frac{r}{e^\alpha})} \frac{w_j(\log(\frac{r}{e^\alpha}) - \log(c_n))}{r^t}$$

$$+\sum_{k(\frac{r}{e^\alpha})<j\leq k(r)} \frac{w_j(\log(r)-\log(c_j))}{r^t}$$

hence

$$\sigma(f,r)\leq \alpha\sum_{n=0}^{k(\frac{r}{e^\alpha})}\frac{w_n}{r^t}+\sum_{j=0}^{k(\frac{r}{e^\alpha})}\frac{w_j(\log(\frac{r}{e^\alpha})-\log(c_n))}{r^t}$$

hence

$$\sigma(f,r)\leq \alpha e^{-t\alpha}\sum_{n=0}^{k(\frac{r}{e^\alpha})}\frac{w_n}{(re^{-\alpha})^t}$$

$$+e^{-t\alpha}\sum_{j=0}^{k(\frac{r}{e^\alpha})}\frac{w_n(\log(\frac{r}{e^\alpha})-\log(c_n))}{(re^{-\alpha})^t}$$

and hence

$$\sigma(f,r)\leq \alpha\psi(f,re^{-\alpha})+e^{-t\alpha}\sigma(f,re^{-\alpha}).$$

Therefore, passing to superior limits on both sides, we have $\sigma(f)\leq \alpha\psi(f)+e^{-t\alpha}\sigma(f)$ and hence

$$\sigma(f)(\frac{1-e^{-t\alpha}}{\alpha})\leq \psi(f).$$

That holds for all $\alpha > 0$, hence we have $\sup_{\alpha>0} \frac{1-e^{-t\alpha}}{\alpha} = t$, and therefore we have $t\sigma(f) \leq \psi(f)$, which proves the inequality:

$$\rho(f)\sigma(f) \leq \psi(f).$$

We will now show the upper bounds of ψ in various cases. First we have

$$\sigma(f,r) = \sum_{n=0}^{k(r)} \frac{w_n(\log(r) - \log(\frac{r}{e^\alpha})) + (\log(\frac{r}{e^\alpha}) - \log(c_n))}{r^t}$$

$$\geq \left(e^{-t\alpha}\right) \frac{\sum_{n=0}^{k(\frac{r}{e^\alpha})} w_n(\log(r) - \log(\frac{r}{e^\alpha}))}{(\frac{r}{e^\alpha})^t}$$

$$+\left(e^{-t\alpha}\right) \frac{\sum_{n=0}^{k(\frac{r}{e^\alpha})} w_n(\log(\frac{r}{e^\alpha} - \log(c_n))}{(\frac{r}{e^\alpha})^t}$$

hence

$$(1) \qquad \sigma(f,r) \geq \alpha e^{-t\alpha}\psi(f, \frac{r}{e^\alpha}) + e^{-t\alpha}\sigma(f, \frac{r}{e^\alpha}).$$

Particularly, from (1) we can derive

$$\sigma(f,r) \geq \alpha e^{-t\alpha}\psi(f, \frac{r}{e^\alpha}))$$

and therefore

$$\sigma(f) \geq \alpha e^{-t\alpha}\psi(f) + e^{-t\alpha}\widetilde{\sigma}(f). \tag{2}$$

But now, this holds for every $\alpha > 0$, hence particularly when $\alpha t = 1$, we have

$$\rho(f)\big(e\sigma(f) - \widetilde{\sigma}(f)\big) \geq \psi(f)$$

which is the left hand inequality of the general conclusion.

Now, suppose that $\sigma(f) = \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^t}$, hence

$$\sigma(f) = \lim_{r\to+\infty} \sigma(f,r) = \lim_{r\to+\infty} \sigma(f, re^{-\alpha}),\ (\alpha > 0).$$

Then, for every $\alpha > 0$, we have
$\sigma(f) = \lim_{r\to+\infty} \sigma(f, \frac{r}{e^\alpha})$, therefore $\sigma(f)(\frac{e^{t\alpha}-1}{\alpha}) \geq \psi(f)$ and hence we obtain
$\psi(f) \leq t\sigma(f)$, i.e. $\psi(f) \leq \rho(f)\sigma(f)$.

Now, suppose that

$$\psi(f) = \lim_{r\to+\infty} \sum_{n=0}^{k(r)} \frac{w_n}{r^t} = \lim_{r\to+\infty} \psi(f,r).$$

We can obviously find a sequence $(r_n)_{n\in\mathbb{N}}$ in $]0,+\infty[$ of limit $+\infty$ such that $\sigma(f)=\lim_{n\to+\infty}\sigma(f,r_ne^{-\alpha})$. Then, by (1) we have

$$\sigma(f,r_n)\geq \alpha e^{-t\alpha}\psi(f,\frac{r_n}{e^{\alpha}})+e^{-t\alpha}\sigma(f,\frac{r_n}{e^{\alpha}})$$

hence

$$\limsup_{n\to+\infty}\sigma(f,r_n)\geq \alpha e^{-t\alpha}\psi(f)+e^{-t\alpha}\sigma(f)$$

and hence

$$\sigma(f)\geq \alpha e^{-t\alpha}\psi(f)+e^{-t\alpha}\sigma(f)$$

therefore $\psi(f)\leq\left(\frac{e^{t\alpha}-1}{\alpha}\right)\sigma(f)$. Finally, $\psi(f)\leq\rho(f)\sigma(f)$.

Remark: 1) When neither σ nor ψ are obtained as veritable limits when r tends to $+\infty$, the method does not let us prove that $\psi=\rho\sigma$, the natural conjecture.

2) Concerning the upper bound $\psi(f)\leq\rho(f)(e\sigma(f)-\widetilde{\sigma}(f))$ it is possible to improve a bit this by defining the number $u_0>0$ such that

$e^{u_0}(u_0 - 1) = -\frac{\widetilde{\sigma}(f)}{\sigma(f)}$ and putting $\alpha_0 = \frac{u_0}{\rho(f)}$. Then we have

$$\psi(f) \leq \frac{e^{\rho(f)\alpha_0}\sigma(f) - \widetilde{\sigma}(f)}{\alpha_0}.$$

Corollary 12.1: *Let $f \in \mathcal{A}(\mathbb{K})$ be not identically zero and have finite growth order. Then if $\sigma(f)$ is finite if and only if so is $\psi(f)$.*

Remark: The conclusions of Theorem 12 hold for $\psi(f) = \sigma(f) = +\infty$.

We will now present Example 3 where neither $\psi(f)$ nor $\sigma(f)$ are obtained as limits but only as superior limits: we will show that the equality $\psi(f) = \rho(f)\sigma(f)$ holds again.

Example 3: Let $r_n = 2^n$, $n \in \mathbb{N}$ and let $f \in \mathcal{A}(\mathbb{K})$ have exactly 2^n zeros in $C(0, r_n)$ and satisfy $f(0) = 1$. Then $q(f, r_n) = 2^{n+1} - 1$ $\forall n \in \mathbb{N}$. We can see that the function $h(r)$ defined in $[r_n, r_{n+1}[$ by $h(r) = \dfrac{q(f,r)}{r}$ is decreasing and satisfies $h(r_n) = \dfrac{2^{n+1} - 2}{2^n}$ and $\lim\limits_{r \to r_{n+1}} \dfrac{h(r)}{r} = \dfrac{2^{n+1} - 2}{2^{n+1}}$. Consequently, $\limsup\limits_{r \to +\infty} h(r) = 2$ and $\liminf\limits_{r \to +\infty} h(r) = 1$.

Particularly, by Theorem 4, we have $\rho(f) = 1$ and of course $\psi(f) = 2$. On the other hand, we can show that $\sigma(f) = 2$.

Now, Theorem 12 and Example 3 suggest the following conjecture:

Conjecture 1: *Let $f \in \mathcal{A}^0(\mathbb{K})$ be such that either $\sigma(f) < +\infty$ or $\psi(f) < +\infty$. Then $\psi(f) = \rho(f)\sigma(f)$.*

Although we can't yet prove Conjecture C1, we will show the following enquality:

Now, by Corollary 9.1, we can also state Corollary 12.2:

Corollary 12.2: *Let $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$, with $g,\ h \in \mathcal{A}(\mathbb{K})$ not identically zero and be such that h has finite order of growth and and finite type of growth. Then f' takes every value $b \in \mathbb{K}$ infinitely many times.*

We will now consider derivatives.

Theorem 13: *Let $f \in \mathcal{A}(\mathbb{K})$ be not identically zero. Then $\rho(f) = \rho(f')$.*

Corollary 13.1 *The derivation on $\mathcal{A}(\mathbb{K})$ restricted to the algebra $\mathcal{A}(\mathbb{K}, t)$ (resp. to $\mathcal{A}^0(\mathbb{K})$) provides that algebra with a derivation.*

In complex analysis, it is known that if an entire function f has order $t < +\infty$, then f and f' have same type. We will check that it is the same here.

Theorem 14: *Let $f \in \mathcal{A}(\mathbb{K})$ be not identically zero, of order $t \in]0, +\infty[$. Then $\sigma(f) = \sigma(f')$.*

By Theorems 12, 13, 14 we can now derive Corollary 14.1

Corollary 14.1: *Let $f \in \mathcal{A}^0(\mathbb{K})$ be not identically zero, of order $t < +\infty$. Then*
$\rho(f)\sigma(f) \leq \psi(f') \leq e\rho(f)\sigma(f)$,
$|\psi(f') - \psi(f)|_\infty \leq (e-1)\rho(f)\sigma(f)$ *and*
$\frac{1}{e-1} \leq \frac{\psi(f')}{\psi(f)} \leq e - 1.$

Corollary 14.2: *Let $f \in \mathcal{A}^0(\mathbb{K})$ be not identically zero, of order $t < +\infty$. Then $\psi(f)$ is finite if and only if so is $\psi(f')$.*

Corollary 14.3: *Let $f \in \mathcal{A}^0(\mathbb{K})$ be not identically zero, of order $t < +\infty$. If $\psi(f) = \lim_{r \to +\infty} \frac{q(f,r)}{t}$ or if $\psi(f') = \lim_{r \to +\infty} \frac{q(f',r)}{r^t}$, then $\psi(f) \leq \psi(f')$.*
Moreover the equality $\psi(f') = \psi(f)$ holds in each one of the following four hypotheses:
a) $\psi(f) = \lim_{r \to +\infty} \psi(f,r)$ and $\psi(f') = \lim_{r \to +\infty} \psi(f',r)$,
b) $\psi(f) = \lim_{r \to +\infty} \psi(f,r)$ and $\sigma(f') = \lim_{r \to +\infty} \sigma(f',r)$,
c) $\sigma(f) = \lim_{r \to +\infty} \sigma(f,r)$ and $\sigma(f') = \lim_{r \to +\infty} \sigma(f',r)$,
d) $\sigma(f) = \lim_{r \to +\infty} \sigma(f,r)$ and $\psi(f') = \lim_{r \to +\infty} \psi(f',r)$.

Conjecture 1 suggests and implies the following Conjecture 2:

Conjecture 2: $\psi(f) = \psi(f')\ \forall f \in \mathcal{A}^0(\mathbb{K})$.

Now, by Theorems 13 and 14 we can state Corollary 14.4

Corollary 14.4: *Let $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$ be not identically zero, with $g,\ h \in \mathcal{A}(\mathbb{K})$, having all residues null and such that h has finite order of growth and finite type of growth. Then f takes every value $b \in \mathbb{K}$ infinitely many times.*

Theorem 15: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be transcendental and of same order $t \in [0, +\infty[$. Then for every $\epsilon > 0$,*

$$\limsup_{r \to +\infty} \Big(\frac{r^{\epsilon} q(g,r)}{q(f,r)} \Big) = +\infty.$$

Remark: Comparing the number of zeros of f' to this of f inside a disk is very uneasy. Now, we can give some precisions. By Theorem 14 we can derive Corollary 16.1.

Corollary 15.1: *Let $f \in \mathcal{A}^0(\mathbb{K})$ be not affine. Then for every $\epsilon > 0$, we have*

$$\limsup_{r \to +\infty} \Big(\frac{r^{\epsilon} q(f',r)}{q(f,r)} \Big) = +\infty$$

and

$$\limsup_{r \to +\infty} \Big(\frac{r^{\epsilon} q(f,r)}{q(f',r)} \Big) = +\infty.$$

We can now give a partial solution to a problem that arose in the study of zeros of derivatives of meromorphic functions: given $f \in \mathcal{A}(\mathbb{K})$, is it possible that f' divides f in the algebra $\mathcal{A}(\mathbb{K})$?

Theorem 16: *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$. Suppose that for some number $s > 0$ we have $\limsup_{r\to+\infty} |q(f,r)|r^s > 0$ (where $|q(f,r)|$ is the absolute value of $q(f,r)$ defined on $\mathbb{K}$). Then f' has infinitely many zeros that are not zeros of f.*

Remark: It is possible to deduce the proof of Theorem 14 by using Lemma 1.4 in [3].

Corollary 16.1: *Let $f \in \mathcal{A}^0(\mathbb{K})$. Then f' has infinitely many zeros that are not zeros of f.*

Proof: Indeed, let f be of order t. By Theorem 4 $\limsup_{r\to+\infty} \frac{q(f,r)}{r^t}$ is a finite number and therefore $\limsup_{r\to+\infty} |q(f,r)|r^t > 0$.

Corollary 16.2 *Let $f \in \mathcal{A}^0(\mathbb{K})$. Then f' does not divide f in $\mathcal{A}(\mathbb{K})$.*

Corollary 16.3 is a partial solution for the p-adic Hayman conjecture when $n = 1$, which is not solved yet.

Corollary 16.3 *Let $f \in \mathcal{M}(\mathbb{K})$ be such that*

$$\limsup_{r \to +\infty} |q(\frac{1}{f}, r)| r^s > 0$$

for some $s > 0$. Then ff' has at least one zero.

Proof: Indeed, suppose that ff' has no zero. Then f is of the form $\frac{1}{h}$ with $h \in \mathcal{A}(\mathbb{K})$ and $f' = -\frac{h'}{h^2}$ has no zero, hence every zero of h' is a zero of h, a contradiction to Theorem 17 since
$\limsup_{r \to +\infty} |q(h, r)| r^s > 0$.

Remarks: Concerning complex entire functions, we check that the exponential is of order 1 but is divided by its derivative in the algebra of complex entire functions.

It is also possible to derive Corollary 16.3 from Theorem 1 in the paper by Jean-Paul Bezivin, Kamal Boussaf and me. Indeed, let $g = \frac{1}{f}$. By Theorem 4, $\limsup_{r \to +\infty} \frac{q(f, r)}{r^t}$ is a finite number. Consequently, there exists $c > 0$ such that
$q(f, r) \leq cr^t \ \forall r > 1$ and therefore the number of poles of g in $d(0, r)$ is upper bounded by cr^t whenever $r > 1$. Consequently, we can apply Theorem

8 and hence the meromorphic function g' has infinitely many zeros. Now, suppose that f' divides f in $\mathcal{A}(\mathbb{K})$. Then every zero of f' is a zero of f with an order superior, hence $\frac{f'}{f^2}$ has no zero, a contradiction.

If the residue characteristic of $\mathbb{K}$ is $p \neq 0$, we can easily construct an example of entire function f of infinite order such that f' does not divide f in $\mathcal{A}(\mathbb{K})$. Let $f(x) = \prod_{n=0}^{\infty} (1 - \frac{x}{\alpha_n})^{p^n}$ with $|\alpha_n| = n+1$. We check that $q(f, n+1) = \sum_{k=0}^{n} p^k$ is prime to p for every $n \in \mathbb{N}$. Consequently, Theorem 16 shows that f is not divided by f' in $\mathcal{A}(\mathbb{K})$. On the other hand, fixing $t > 0$, we have

$$\frac{q(f, n+1)}{(n+1)^t} \geq \frac{p^n}{(n+1)^t}$$

hence

$$\limsup_{r \to +\infty} \frac{q(f, r)}{r^t} = +\infty \ \forall t > 0$$

therefore, f is not of finite order.

Theorem 16 suggests the following conjecture:

Conjecture 3 *Given* $f \in \mathcal{A}(\mathbb{K})$ *(other than* $(x-a)^m,\ a \in \mathbb{K},\ m \in \mathbb{N}$*) there exists no* $h \in \mathcal{A}(\mathbb{K})$ *such that* $f = f'h$*.*