Nonlocal modified gravity

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Motivation

Large cosmological observational findings:

- High orbital speeds of galaxies in clusters.(F.Zwicky, 1933)
- High orbital speeds of stars in spiral galaxies. (Vera Rubin, at the end of 1960es)
- Accelerated expansion of the Universe. (1998)

Problem solving approaches

There are two problem solving approaches:

- Dark matter and energy
- Modification of Einstein theory of gravity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},\ c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.

Dark matter and energy

- If Einstein theory of gravity can be applied to the whole Universe then the Universe contains about 4.9% of ordinary matter, 26.8% of dark matter and 68.3% of dark energy.
- It means that 95.1% of total matter, or energy, represents dark side of the Universe, which nature is unknown.
- Dark matter is responsible for orbital speeds in galaxies, and dark energy is responsible for accelerated expansion of the Universe.

Modification of Einstein theory of gravity

Motivation for modification of Einstein theory of gravity

- The validity of General Relativity on cosmological scale is not confirmed.
- Dark matter and dark energy are not yet detected in the laboratory experiments.
- Another cosmological problem is related to the Big Bang singularity. Namely, under rather general conditions, general relativity yields cosmological solutions with zero size of the universe at its beginning, what means an infinite matter density.
- Note that when physical theory contains singularity, it is not valid in the vicinity of singularity and must be appropriately modified.

Approaches to modification of Einstein theory of gravity

There are different approaches to modification of Einstein theory of gravity.

- Einstein General Theory of Relativity

From action $S = \int (\frac{R}{16\pi G} - L_m - 2\Lambda)\sqrt{-g}d^4x$ using variational methods we get field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},\ c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.
Currently there are mainly two approaches:

- f(R) Modified Gravity
- Nonlocal Gravity

Nonlocal Modified Gravity

Nonlocal gravity is a modification of Einstein general relativity in such way that Einstein-Hilbert action contains a function f(□, R). Our action is given by

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + R^p\mathcal{F}(\Box)R^q\Big)$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$ and C is a constant.

Equations of motion

Let us introduce the following axillary actions

$$S_0 = \int (R - 2\Lambda)\sqrt{-\mathrm{g}}\; d^4x,$$
$$S_1 = \int R^p \mathcal{F}(\Box)\mathcal{R}^q \sqrt{-\mathrm{g}}\; d^4x.$$

Then the variation of the action S is

$$\delta S = \frac{1}{16\pi G}\delta S_0 + \delta S_1.$$

Also we assume that the variation of the metric coefficients and their first derivatives vanish on the border of the manyfold M, i.e. $\delta g_{\mu\nu}|_{\partial M} = 0$, $\delta \partial_\lambda g_{\mu\nu}|_{\partial M} = 0$.

Variation of action S_0

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Lemma

On any manfold M we have

$$\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\ d^4x = 0$$

.

Lemma

Variation of the action S_0 is

$$\delta S_0 = \int G_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x + \Lambda\int g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x,$$

Variation of the action S_1

Lemma

For any scalar function $h(R)$ the following equation holds

$$\int h\delta R\sqrt{-g}\, d^4x = \int \left(hR_{\mu\nu} + g_{\mu\nu}\Box h - \nabla_\mu\nabla_\nu h\right)\delta g^{\mu\nu}\sqrt{-g}\, d^4x.$$

Proof. It can be shown that

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}.$$

Odavde sledi da za bilo koju skalarnu funkciju h vazhi Moreover, for any scalar function $h(R)$ we have the following identity

$$\int h\delta R\sqrt{-g}\, d^4x$$

$$= \int \left(hR_{\mu\nu}\delta g^{\mu\nu} + hg_{\mu\nu}\Box\delta g^{\mu\nu} - h\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\sqrt{-g}\, d^4x.$$

Last two terms can be transformed using the following two identities

$$\int hg_{\mu\nu}\Box\delta g^{\mu\nu}\sqrt{-g}\;d^4x = \int g_{\mu\nu}\Box h\delta g^{\mu\nu}\sqrt{-g}\;d^4x,$$

$$\int h\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\sqrt{-g}\;d^4x = \int \nabla_\mu\nabla_\nu h\;\delta g^{\mu\nu}\sqrt{-g}\;d^4x.$$

To prove the first identity we use Stokes' theorem in the following way

$$\begin{aligned}\int hg_{\mu\nu}\Box\delta g^{\mu\nu}\sqrt{-g}\;d^4x &= \int hg_{\mu\nu}\nabla_\alpha\nabla^\alpha\delta g^{\mu\nu}\sqrt{-g}\;d^4x\\ &= -\int \nabla_\alpha(hg_{\mu\nu})\nabla^\alpha\delta g^{\mu\nu}\sqrt{-g}\;d^4x\\ &= \int \nabla^\alpha\nabla_\alpha(hg_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g}\;d^4x\\ &= \int g_{\mu\nu}\nabla^\alpha\nabla_\alpha h\;\delta g^{\mu\nu}\sqrt{-g}\;d^4x\\ &= \int g_{\mu\nu}\Box h\;\delta g^{\mu\nu}\sqrt{-g}\;d^4x.\end{aligned}$$

To obtain the second identity, we introduce the following vector

$$N^\mu = h\nabla_\nu \delta g^{\mu\nu} - \nabla_\nu h \delta g^{\mu\nu}.$$

From the above expression, one obtains

$$\begin{aligned}
\nabla_\mu N^\mu &= \nabla_\mu (h\nabla_\nu \delta g^{\mu\nu} - \nabla_\nu h \delta g^{\mu\nu}) \\
&= \nabla_\mu h \nabla_\nu \delta g^{\mu\nu} + h\nabla_\mu \nabla_\nu \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu h \, \delta g^{\mu\nu} - \nabla_\nu h \nabla_\mu \delta g^{\mu\nu} \\
&= h\nabla_\mu \nabla_\nu \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu h \, \delta g^{\mu\nu}.
\end{aligned}$$

Integrating divergence $\nabla_\mu N^\mu$ yields

$$\int \nabla_\mu N^\mu \sqrt{-g} \, d^4x = \int_{\partial M} N^\mu n_\mu d\partial M = 0,$$

where n_μ is a unit normal. As N^μ vanishes on the border ∂M, we conclude that the last term is zero, which completes the proof.

Variation of the action S_1

Lemma

Let $\theta(R)$ and $\psi(R)$ scalar functions such that $\delta\psi|_{\partial M} = 0$. Then, it follows

$$\begin{aligned}\int \theta\delta\Box\psi\sqrt{-g}\, d^4x = &\frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\,\partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\,\sqrt{-g}\, d^4x \\ &- \int \partial_\mu\theta\,\partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x + \int \Box\theta\,\delta\psi\sqrt{-g}\, d^4x \\ &+ \frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x.\end{aligned}$$

Variation of the action S_1

Proof.

$$\int \theta \delta \Box \psi \sqrt{-g}\, d^4x = \int \theta \partial_\alpha \delta(\sqrt{-g} g^{\alpha\beta} \partial_\beta \psi)\, d^4x$$
$$+ \int \theta \delta \left(\frac{1}{\sqrt{-g}} \right) \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta \psi) \sqrt{-g}\, d^4x$$
$$= \int \partial_\alpha (\theta \delta (\sqrt{-g} g^{\alpha\beta} \partial_\beta \psi))\, d^4x - \int \partial_\alpha \theta\, \delta (\sqrt{-g} g^{\alpha\beta} \partial_\beta \psi)\, d^4x$$
$$+ \frac{1}{2} \int \theta g_{\mu\nu} \Box \psi \delta g^{\mu\nu} \sqrt{-g}\, d^4x.$$

After a short calculation one obtains
$\int \partial_\alpha (\theta \delta (\sqrt{-g} g^{\alpha\beta} \partial_\beta \psi))\, d^4x = 0.$

Then, it follows

$$\int \theta\delta\Box\psi\sqrt{-g}\; d^4x$$

$$= -\int g^{\alpha\beta}\partial_\alpha\theta\; \partial_\beta\psi\delta(\sqrt{-g})\; d^4x - \int \partial_\alpha\theta\; \partial_\beta\psi\delta g^{\alpha\beta}\sqrt{-g}\; d^4x$$

$$- \int g^{\alpha\beta}\sqrt{-g}\partial_\alpha\theta\; \partial_\beta\delta\psi\; d^4x + \frac{1}{2}\int \theta g_{\mu\nu}\Box\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x$$

$$= \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\; \partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\; \sqrt{-g}\; d^4x - \int \partial_\mu\theta\; \partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x$$

$$- \int \partial_\beta(g^{\alpha\beta}\sqrt{-g}\partial_\alpha\theta\; \delta\psi)\; d^4x + \int \partial_\beta(g^{\alpha\beta}\sqrt{-g}\partial_\alpha\theta)\; \delta\psi\; d^4x$$

$$+ \frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x$$

$$= \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\; \partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\; \sqrt{-g}\; d^4x - \int \partial_\mu\theta\; \partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x$$

$$+ \int \Box\theta\; \delta\psi\sqrt{-g}\; d^4x + \frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x.$$

At the end we obtain

$$\int \theta\delta\Box\psi\sqrt{-g}\ d^4x = \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\ \partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\ \sqrt{-g}\ d^4x - \int \partial_\mu\theta\ \partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\ d^4x + \int \Box\theta\ \delta\psi\sqrt{-g}\ d^4x + \frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\ d^4x.$$

Variation of the action S_1

At the end we can prove the following lemma

Lemma

Variation of the naction S_1 is given by

$$\delta S_1 = -\frac{1}{2}\int g_{\mu\nu} R^p \mathcal{F}(\Box) R^q \delta g^{\mu\nu}\sqrt{-g}\, d^4x$$

$$+ \int (R_{\mu\nu} W - K_{\mu\nu} W)\, \delta g^{\mu\nu}\sqrt{-g}\, d^4x$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int \Big(g_{\mu\nu}\left(\partial^\alpha \Box^l R^p \partial_\alpha \Box^{n-1-l} R^q + \Box^l R^p \Box^{n-l} R^q\right)$$

$$- 2\partial_\mu \Box^l R^p \partial_\nu \Box^{n-1-l} R^q \Big)\delta g^{\mu\nu}\sqrt{-g}\, d^4x,$$

Where, $K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$,
$W = pR^{p-1}\mathcal{F}(\Box)R^q + qR^{q-1}\mathcal{F}(\Box)R^p$, *and* $'$ *denotes derivation over* R.

Variation of the action S and EOM

Theorem

Variation of the action S vanishes iff

$$\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} + -\frac{1}{2} g_{\mu\nu} R^p \mathcal{F}(\Box) R^q + (R_{\mu\nu} W - K_{\mu\nu} W)$$
$$+ \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Big(g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Box^l R^p \partial_\beta \Box^{n-1-l} R^q$$
$$- 2\partial_\mu \Box^l R^p \partial_\nu \Box^{n-1-l} R^q + g_{\mu\nu} \Box^l R^p \Box^{n-l} R^q \Big) = 0, \qquad (1)$$

Trace and 00-component of EOM

Theorem

Assume that we use FRW metric. Then there are two linearly independent equations. The most convinent choice is:

$$\frac{4\Lambda - R}{16\pi G} + C\Big(-2R^P \mathcal{F}(\Box) R^q + (RW + 3\Box W) + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\partial_\mu \Box^l R^p \partial^\mu \Box^{n-1-l} R^q + 2\Box^l R^p \Box^{n-l} R^q\right)\Big) = 0,$$

$$\frac{G_{00} + \Lambda g_{00}}{16\pi G} + C\Big(-\frac{1}{2} g_{00} R^p \mathcal{F}(\Box) R^q + (R_{00} W - K_{00} W) + \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(g_{00} g^{\alpha\beta} \partial_\alpha \Box^l R^p \partial_\beta \Box^{n-1-l} R^q - 2\partial_0 \Box^l R^p \partial_0 \Box^{n-1-l} R^q + g_{00} \Box^l R^p \Box^{n-l} R^q\big)\Big) = 0.$$

FLRW metric

In case of Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big), \ k \in \{-1, 0, 1\}.$$

there are two linearly independent equations

$$\frac{4\Lambda - R}{16\pi G} - 2R^p \mathcal{F}(\Box) R^q + (RW + 3\Box W) + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Big(\partial_\mu \Box^l R^p \partial^\mu \Box^{n-1-l} R^q + 2\Box^l R^p \Box^{n-l} R^q\Big) = 0,$$

$$\frac{G_{00} + \Lambda g_{00}}{16\pi G} - \frac{1}{2} g_{00} R^p \mathcal{F}(\Box) R^q + (R_{00} W - K_{00} W) + \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Big(g_{00} g^{\alpha\beta} \partial_\alpha \Box^l R^p \partial_\beta \Box^{n-1-l} R^q - 2\partial_0 \Box^l R^p \partial_0 \Box^{n-1-l} R^q + g_{00} \Box^l R^p \Box^{n-l} R^q\Big) = 0.$$