Stochastic Methods for Finite Approximations in a non-Archimedean Setting

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Outline

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Motivation for using stochastic methods

- Expand the non-Archimedean toolbox.
- Stochastic methods may in some cases give stronger results.

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Setting

Schrödinger operator (cfr.[VVZ94])

- $H = P^\alpha + V$, acting on $L^2(X^d)$ where $X = \mathbf{R}$ or $X = K$, a local field. For simplicity we will work in dimension 1 ($d = 1$).
- $P = \mathcal{F}^{-1} Q \mathcal{F}$, $Q =$ multiplication by the absolute value of the coordinate: $(Qf)(x) = |x| f(x)$.
- α : a positive real number. If $X = \mathbf{R}$ and $\alpha = 2$, we recover the Laplacian: $P^2 = -\Delta$.
- V (potential): multiplication by a continuous function v which goes to infinity at infinity, implying discrete spectrum for H.

Setting (cont'd)

For suitably defined finite models (to be specified later) it has been shown that the spectral data at the finite level converge in a strong sense to the spectral data of the infinite model.

Conventional quantum system over the reals

[DVV94]:

- Gave two proofs: A conventional functional analytic one, and a stochastic one.
- The stochastic proof gave a somewhat stronger result, at the expense of a mild extra assumption.

Quantum system over a local field

- [BD15]: Proved finite approximation theorem for a quantum system over a general local field. Proof was functional analytic.
- In this talk: Stochastic proof of same.

Local fields

- K: a local field, i.e., a non-discrete, totally disconnected, locally compact field.
- Two main types of local fields:
  - char $K = 0$: K is a finite extension of $\mathbf{Q}_p$ for some p.
  - char $K > 0$: K is isomorphic to the field of Laurent series over a finite field $\mathbf{F}_q$, where $q = p^f$, $p =$ char K.
- $|\cdot|$: canonical absolute value, induced by the Haar measure. It defines the topology, and is non-Archimedean.
- $O = \{x \in K : |x| \leq 1\}$: a compact sub-ring of K called the ring of integers. It is a discrete valuation ring, i.e., a principal ideal domain with a unique maximal ideal.
- $P = \{x \in K : |x| < 1\}$: the unique maximal ideal of O, called the prime ideal. We have $P = \beta O$ for some $\beta \in O$. Such a β is called a *uniformizer*.
- O/P is a finite field. If $q = p^f$ is the number of elements in O/P, then $|\beta| = 1/q$ for any uniformizer β.

Local fields (cont'd)

- If S is a complete set of representatives for O/P, every $x \in K$ can be written uniquely in the form

$$x = \beta^{-m}(x_0 + x_1\beta + x_2\beta^2 + \cdots),$$

where $m \in \mathbf{Z}$, $x_j \in S$, $x_0 \notin P$. With x written in this form, we have $|x| = q^m$. So range$(|\cdot|) = \{q^N \colon N \in \mathbf{Z}\}$.

Haar measure, characters and Fourier transform

Fix a Haar measure μ such that $\mu(O) = 1$, and define the Fourier transform $\mathcal{F}$ by

$$(\mathcal{F}f)(\xi) = \int_K f(x)\chi(-x\xi)\,dx$$

for a suitably chosen additive character χ. For our setup it will be essential to work with a character of rank 0.[1]

[1] $\text{rank}(\chi) = \max\{r \in \mathbf{Z} : \chi|_{B_r} \equiv 1\}$, $B_r = \{x \in K : |x| \leq q^r\}$.

Character of rank zero: char $K = 0$

$$\chi(x) = \chi_p\left(\mathrm{Tr}_{K/\mathbf{Q}_p}(\beta^{-d}x)\right), \quad x \in K,$$

where

- χ_p is the canonical character on $\mathbf{Q}_p$—i.e., $\chi_p(x) = \exp(2\pi i\{x\})$, $\{x\}$ = fractional part of x.
- $\mathrm{Tr}_{K/\mathbf{Q}_p} : K \to \mathbf{Q}_p$ is the trace function associated with the extension $K/\mathbf{Q}_p$.
- β is a uniformizer as defined above.
- d is the *exponent of the different* of the extension $K/\mathbf{Q}_p$. It is the largest integer d such that $\mathrm{Tr}_{K/\mathbf{Q}_p}(x) \in \mathbf{Z}_p$ for all x with $|x| \leq q^d$ (note that $d \geq 0$ since $\mathrm{Tr}_{K/\mathbf{Q}_p} : O \to \mathbf{Z}_p$).

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Character of rank zero: char $K > 0$

In this case we may identify K with the field $\mathbf{F}_q((t))$ of Laurent series in the indeterminate t with coefficients from the finite field $\mathbf{F}_q$, $q = p^f$, consisting of elements of the form $x = \sum_{i=m}^{\infty} x_i t^i$, $x_i \in \mathbf{F}_q$, $m \in \mathbf{Z}$. Let η denote the canonical character on $\mathbf{F}_q$, i.e., $\eta(x) = \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x)\right)$, and define

$$\chi(x) = \eta(x_{-1}),$$

where x_{-1} refers to the expansion $x = \sum_{i=m}^{\infty} x_i t^i$. Then χ is a rank zero character on $K = \mathbf{F}_q((t))$.

Finite model

- $B_n = \beta^{-n}O$ = ball of radius q^n: an open additive subgroup of K.
- $G_n = B_n/B_{-n}$: a finite group with q^{2n} elements ($n \geq 0$).[2]
- Each element of G_n has a unique representative of the form

$$a_{-n}\beta^{-n} + a_{-n+1}\beta^{-n+1} + \cdots + a_{-1}\beta^{-1} + a_0 + a_1\beta + \cdots \\ + a_{n-2}\beta^{n-2} + a_{n-1}\beta^{n-1}.$$

  We denote the set of these representatives by X_n, and give it the group structure inherited from G_n.
- Haar measure μ_n on G_n:
  $\mu_n(\{x + H_n\}) = \mu(x + H_n) = \mu(H_n) = q^{-n}$, $\{x + H_n\} \in G_n$.
  So each point $\{x + H_n\}$ of G_n has mass q^{-n}, and the total mass of G_n is $q^{2n} \cdot q^{-n} = q^n$.

[2] For convenience we often write $H_n = B_{-n}$; so for instance we have $G_n = H_{-n}/H_n$.

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Finite model (cont'd)

- L^2-isometric imbedding $L^2(G_n) \to L^2(K)$:

$$\mathbf{1}_{\{x+H_n\}} \in L^2(G_n) \mapsto \mathbf{1}_{x+H_n} \in L^2(K).$$

  An operator on $L^2(G_n)$ is regarded as an operator on $L^2(K)$ via this imbedding, by setting it equal to 0 on the orthogonal complement of the image of $L^2(G_n)$.

Important subspaces of $L^2(K)$

- $\mathcal{C}_n = \{f \in L^2(K) | \operatorname{supp}(f) \subset B_n\}$. The corresponding orthogonal projection is denoted by C_n and is given by: $C_n f = \mathbf{1}_{B_n} f$.
- $\mathcal{S}_n = \{f \in L^2(K) | f$ is locally constant of index $\leq q^{-n}\}$. The corresponding orthogonal projection is denoted by S_n and is given by:
  $(S_n f)(x) = q^n \int_{H_n} f(x+y)\,dy = \frac{1}{\mu(H_n)} \int_{H_n} f(x+y)\,dy =$ $\operatorname{ave}(f, n, x)$, where we have introduced the notation $\operatorname{ave}(f, n, x)$ for the average value of f over $x + H_n$.
- $\mathcal{D}_n = \mathcal{C}_n \cap \mathcal{S}_n$. The corresponding orthogonal projection is denoted by D_n.

$L^2(G_n)$ is mapped onto $\mathcal{D}_n$ via the isometric imbedding mentioned above. Thus $L^2(G_n)$ can be thought of as the set of functions on K which have support in B_n and which are invariant under translation by elements of $H_n (= B_{-n})$.

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Commutation relations

$$D_n = C_n S_n = S_n C_n$$

$$\mathcal{F}\mathcal{C}_n = \mathcal{S}_n, \quad \mathcal{F}\mathcal{S}_n = \mathcal{C}_n, \quad \mathcal{F}\mathcal{D}_n = \mathcal{D}_n$$

$$\mathcal{F} C_n = S_n \mathcal{F}, \quad \mathcal{F} S_n = C_n \mathcal{F}, \quad \mathcal{F} D_n = D_n \mathcal{F}$$

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Fourier transform at the finite level

Let as before χ be a rank zero character on K. The bi-character $(x, y) \mapsto \chi(xy)$ descends to a non-degenerate bi-character on $G_n = B_n/B_{-n}$, thus the natural choice for an L^2-isometric Fourier transform on $X_n \cong G_n$ is

$$(\mathcal{F}_n f)(x) = \frac{1}{\sqrt{|X_n|}} \sum_{y \in X_n} f(y)\chi(-xy)$$
$$= q^{-n} \sum_{y \in X_n} f(y)\chi(-xy), \quad x \in X_n, \quad f \in L^2(X_n).$$

Crucial fact:
The Fourier transform $\mathcal{F}$ on K descends to the Fourier transform $\mathcal{F}_n$ on X_n:

$$\mathcal{F}|_{\mathcal{D}_n} = \mathcal{F}_n, \text{ i.e.}, \mathcal{F}_n = \mathcal{F} D_n = D_n \mathcal{F}.$$

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Dynamical operators at the finite level

The finite operators are obtained through compression by the projection D_n:

$$V_n = D_n V D_n, \quad Q_n = D_n Q D_n, \quad P_n = D_n P D_n = \mathcal{F}_n^{-1} Q_n \mathcal{F}_n$$

We have, for $f \in L^2(X_n)$:

$$(V_n f)(x) = v_n(x) f(x), \quad v_n(x) = \frac{1}{\mu(H_n)}[\int_{x+H_n} v(x+h)\, dh]$$

$$(Q_n f)(x) = q_n(x) f(x), \quad q_n(x) = \frac{1}{\mu(H_n)}[\int_{x+H_n} |h|\, dh]$$

$$= \begin{cases} |x|, & |x| > q^{-n} \\ \text{ave}(|x|, n, 0), & |x| \leq q^{-n} \end{cases}$$

$$H_n = P_n^\alpha + V_n = \mathcal{F}_n^{-1} Q_n^\alpha \mathcal{F}_n + V_n \text{ (finite Hamiltonian.)}$$

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Finite approximation theorem

Theorem (Theorem 4.1 in [BD15])

1. *If J is a compact subset of $[0, \infty)$ with $J \cap \sigma(H) = \emptyset$, then $J \cap \sigma(H_n) = \emptyset$ for large n.*
2. *If $\lambda \in \sigma(H)$, there exists a sequence (λ_n) with $\lambda_n \in \sigma(H_n)$ such that $\lambda_n \to \lambda$. Further, if J is a compact neighborhood of an eigenvalue $\lambda \in \sigma(H)$, not containing any other eigenvalues of H, then any sequence λ_n with $\lambda_n \in \sigma(H_n) \cap J$ converges to λ.*
3. *Let λ and J be as in (2). Then $\dim P^{H_n}(J) = \dim P^H(J)$ for large n, and for each orthonormal basis $\{e_1, \ldots, e_m\}$ for $r\left(P^H(J)\right)$ there is, for each n, an orthonormal basis $\{e_1^n, \ldots, e_m^n\}$ for $r\left(P^{H_n}(J)\right)$ such that $\lim_{n\to\infty} e_i^n = e_i$, $i = 1, \ldots, m$.*

Stochastics at the finite level I

For stochastics over a general local field, see [Koc01, Ch. 4.2] and [Var97].
If as in [Var97] we define

$$\rho_t = \mathcal{F}^{-1}\sigma_t, \quad \sigma_t(x) = e^{-t|x|^\alpha},$$

then ρ_t is a density function, and $e^{-tP^\alpha} f = \rho_t * f$.
At the finite level we want a density $\rho_{t,n}$ such that $e^{-tP_n^\alpha} f = \rho_{t,n} * f$. So we try with

$$\rho_{t,n} = \mathcal{F}_n^{-1}\sigma_{t,n} \quad \sigma_{t,n}(x) = c_{t,n} e^{-t\, r_n(x)^\alpha}$$

where

$$r_n(x) = \begin{cases} |x|, & |x| > q^{-n} \\ \mathrm{ave}(|x|, n, 0), & |x| \leq q^{-n} \end{cases},$$

and $c_{t,n}$ is a positive number, adjusted so that $\sigma_{t,n}(0) = 1$.
The following can be shown (cfr. Erik Bakken's talk):

Stochastics at the finite level II

- For each $a \in K$ and $T > 0$ the density $\rho_{t,n}$ gives rise to a probability measure $\mathbf{P}_a^n$ on the Skorokhod space $D([0,T],K)$ of paths in K starting at a.
  If $a \in G_n$, the measure $\mathbf{P}_a^n$ gives full measure to the Skorokhod space $D([0,T],G_n)$ of paths in G_n starting at a.
- For $a, b \in G_n$ the density $\rho_{t,n}$ gives rise to a probability measure $\mathbf{P}_{a,b,T}^n$ on the Skorokhod space $D([0,T],K)$ which gives full measure to the paths in G_n which start at a and arrive at b at time T.
- If $a_n, b_n \in G_n$, $a, b \in K$, and $(a_n, b_n) \to (a, b)$, then $\mathbf{P}_{a_n}^n \Longrightarrow \mathbf{P}_a$ and $\mathbf{P}_{a_n,b_n,T}^n \Longrightarrow \mathbf{P}_{a,b,T}$ as $n \to \infty$. Here $\Longrightarrow$ denotes weak convergence of measures, and $\mathbf{P}_a$ (resp. $\mathbf{P}_{a,b,T}$) is the unconditioned (resp. conditioned) measure on $D([0,T],K)$ induced by the density ρ_t. If a and b vary in a compact set, the convergence is uniform with respect to a, b.

Feynman-Kac at the finite level

Theorem

For $a, b \in G_n$ we have

$$e^{-t\,H_n}(a,b) = [\int_{D([0,t],K)} e^{-\int_0^t v_n((\omega(s))\,ds} d\mathbf{P}^n_{a,b,t}(\omega)] \cdot \rho_{t,n}(b-a) q^{-n}\,,$$

where $e^{-t\,H_n}(a,b)$ is the (a,b)-entry of the matrix operator $e^{-t\,H_n}$ on the finite dimensional space $L^2(G_n)$, the matrix being with the respect to the canonical basis of $L^2(G_n)$.

Trace convergence I

Theorem

$$\mathrm{Tr}(e^{-t\,H_n}) \to \mathrm{Tr}(e^{-t\,H}).$$

Some elements from the proof:

- First show that for any fixed m

$$\sum_{j \in B_m \cap G_n} e^{-tH_n}(j,j) \to \int_{B_m} K_t(x,x)\,dx$$

where
$K_t(x,y) = \int_{D([0,t],K)} e^{-\int_0^t v(\omega(s))\,ds} dP_{x,y,t}(\omega) \cdot \rho_t(y-x)$
and

$$(e^{tH} f)(x) = \int_K K_t(x,y) f(y)\,dy \quad \text{(Feynman-Kac for } K\text{)}$$

Trace convergence II

- To control what happens outside a given ball, we show that for any $k > 0$ and $j \in G_n$ with $|j| = q^m$ there exists a constant B_k, independent of m and n, such that

$$e^{-tH_n}(j,j) \leq q^{-n} B_k \left(e^{-\frac{t}{2} v^*(\beta - m)} + \frac{1}{q^{mk}} \right)$$

  for all m, n. Here $v^*(x) = \inf_{|y|=|x|} v(y)$.

- Conclude that e^{-tH} is of trace class and that

$$\lim_{n\to\infty} \sum_{j \in G_n} e^{-tH_n}(j,j) \to \int_K K_t(x,x)\,dx.$$

  The left hand side is equal to $\lim_{n\to\infty} \mathrm{Tr}(e^{-tH_n})$, and the right hand side is by Mercer´s Theorem equal to $\mathrm{Tr}(e^{-tH})$. The proof required the extra condition that $\frac{\ln(|x|)}{|v(x)|} \to 0$ as $|x| \to \infty$, and that $\alpha > 1$.

Convergence of spectrum I

Theorem

$$||e^{-t\,H_n} - e^{-t\,H}||_1 \to 0.$$

Sketch of proof: First show that $||e^{-t\,H_n} - e^{-t\,H}||_2 \to 0$ (Hilbert-Schmidt norm). This will follow if we can prove:

$$||e^{-tH_n}||_2 \to ||e^{-tH}||_2 \,,$$

$$\text{and} \quad \langle e^{-tH_n}, L\rangle \to \langle e^{-tH}, L\rangle$$

for all Hilbert-Schmidt operators L. For the first fact we have:

$$||e^{-tH_n}||_2^2 = \mathrm{Tr}(e^{-2tH_n}) \to \mathrm{Tr}(e^{-2tH}) = ||e^{-tH}||_2^2$$

and the second fact follows easily by first showing it for operators L of finite rank.

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Convergence of spectrum II

To go from convergence in Hilbert-Schmidt norm to convergence in trace norm, we use the inequality

$$||A^2 - B^2||_1 \leq ||(A+B)||_2 \cdot ||(A-B)||_2 + 2||B||_2 \cdot ||(A-B)||_2$$

which follows from

$$A^2 - B^2 = (A+B)(A-B) + (A-B)B - B(A-B).$$

With $A = e^{-\frac{t}{2}H_n}$ and $B = e^{-\frac{t}{2}H}$, we get

$$||e^{-tH_n} - e^{-tH}||_1 \leq ||(e^{-\frac{t}{2}H_n} + e^{-\frac{t}{2}H})||_2 \cdot ||(e^{-\frac{t}{2}H_n} - e^{-\frac{t}{2}H})||_2 \\ + 2||e^{-\frac{t}{2}H}||_2 \cdot ||(e^{-\frac{t}{2}H_n} - e^{-\frac{t}{2}H})||_2$$

which goes to 0 as $n \to \infty$.

End of proof.

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Convergence of spectrum III

- Convergence in trace norm implies convergence in operator norm which gives convergence of eigenvalues and eigenfunctions, and we have reproved the finite approximation theorem (frame 19) by stochastic methods.
- To reap the benefits from using stochastic methods, we will now show that we actually have uniform convergence on compacta for the eigenfunctions.

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Uniform convergence on compacta of the eigenfunctions I

Lemma

For each $t > 0$, there exists a constant $C = C(t)$ such that for any $h \in L^2(K)$ and any n,

$$||e^{-tH_n}h||_\infty \leq C||h||_2, \qquad ||e^{-tH}h||_\infty \leq C||h||_2.$$

Lemma

Fix $t > 0$. Then for each $h \in L^2(\mathbb{Q}_p)$,

$$e^{-tH_n}h \to e^{-tH}h$$

uniformly on compacta.

Uniform convergence on compacta of the eigenfunctions II

Theorem

Let $f_{n,j}$ and f_j be the eigenfunctions of H_n and H corresponding to the eigenvalues $\lambda_{n,j}$ and λ_j respectively. Assume that $\lambda_{n,j}$ converges to λ_j and that $f_{n,j}$ converges to f_j in $L^2(\mathbb{Q}_p)$. Then

$$f_{n,j} \to f_j$$

uniformly on compacta.

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Numerical results for the Schrödinger operator over $\mathbf{Q}_3[\sqrt{3}]$

The following pages are extracted from the article "Finite Approximations of Physical Models over Local Fields" (joint work with Erik M. Bakken) which is due to appear in the journal "p-Adic Numbers, Ultrametric Analysis, and Applications". It is also available on the arXiv: http://arxiv.org/abs/1509.04175

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5. Numerical investigation of the Schrödinger operator over $\mathbf{Q}_3[\sqrt{3}]$

5.1. **Overview.** In [VVZ94, Ch. 3, Section XII] a detailed analysis was carried out on the spectrum of the p-adic Schrödinger operator, and in [Koc01, Ch. 3] a similar analysis was performed on the Schrödinger operator over a general local field.

Let as before $H = P^\alpha + V$ denote the Schrödinger operator over a local field K. The eigenfunctions of H can be divided into two main types, corresponding to two complementary subspaces of $L^2(K)$: those which are supported on a single spherical shell (which we shall call shell functions), and those which are radial[4]. Of these, only the shell functions are completely understood: They belong to eigenvalues which can be determined from Diophantine equations, and there are explicit formulae for them. For radial eigenfunctions no such explicit formulae seem to be known.

In this numerical study we specialize to the case of the Schrödinger operator $H = \frac{1}{2}(P^2 + Q^2)$ of the harmonic oscillator over the local field $\mathbf{Q}_3[\sqrt{3}]$, which is a quadratic and totally ramified extension of $\mathbf{Q}_3$. We were interested in the following questions:

- Do eigenfunctions of both types (shell functions and radial functions) show up already at the finite level?
- Is there good agreement between the theoretical and numerical eigenvalues?
- Is there good agreement between the theoretical and numerical eigenfunctions?
- Are multiplicities correct?

The answer to all these questions was 'yes'. To illustrate this, we sum up some of the results in Table 1.

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[4]With notation as in [VVZ94, Koc01], the set of shell functions comprises all the type I functions plus the shell functions of type II; the radial functions are all of type II.

5.2. **More details about the numerical experiment.** The extension $\mathbf{Q}_3[\sqrt{3}]/\mathbf{Q}_3$ is totally ramified, so with notation as in section 2 we have $e = 2$, and hence $f = 1$ since $ef = [\mathbf{Q}_3[\sqrt{3}] : \mathbf{Q}_3] = 2$. Further, from $q = p^f$ follows $q = p = 3$, and as uniformizer we can take $\beta = \sqrt{3}$, hence $|\beta| = 1/q = 1/3$. For the exponent of the different we have $d = 1$, so the character χ defined in subsection 2.1 becomes $\chi(x) = \exp\left(2\pi i\{\mathrm{Tr}_{\mathbf{Q}_3[\sqrt{3}]/\mathbf{Q}_3}(\sqrt{3}^{-1}x)\}\right)$, $x \in \mathbf{Q}_3[\sqrt{3}]$.

For the finite model we did experiments with $n = 1, 2, 3, 4$, so we were working with finite grids of sizes $|X_1| = 9$, $|X_2| = 9^2 = 81$, $|X_3| = 9^3 = 729$, and $|X_4| = 9^4 = 6561$, respectively. Of particular interest to us was how the eigenfunctions came out: Would they clearly exhibit characteristics as shell functions or radial functions? They did. To illustrate this we give in Table 2 an excerpt from the value tables of three eigenfunctions: one is radial, one is a linear combination of two shell functions, and one is a pure shell function. We also wanted to compare our numerically computed eigenfunctions to the theoretical ones (evaluated on the grid). To do this, we measured the distance from each of the former to the linear span of the latter. Up to machine accuracy (10^{-16}), the distance came out as zero. We find this quite remarkable.

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Appendix A. Tables for numerical eigenvalues and eigenfunctions

The tables in this section should be self-explanatory[5]. The data are taken from a computer run with $n = 2$ (i.e., 81 points in the finite grid). Each of the functions in Table 2 is represented with 28 values, with values coming from each of the 5 shells which occur for $n = 2$.

[5]In the estimate for the lowest eigenvalue in Table 1 (first entry in column 1) we are assuming that the estimate given in [VVZ94, p. 190] is valid also in our setting. We havent checked this in detail, but there are strong indications that it is true.

Table 1. Numerical approximations to the spectral data of $H = \frac{1}{2}(P^2 + Q^2)$ over $\mathbf{Q}_3[\sqrt{3}]$.

Theoretical eigenvalue	Numerical eigenvalue	Theoretical multiplicity	Numerical multiplicity	Type of eigenfunction	Comment
$0 < \lambda_0 < 9/13 \approx 0.6923$	0.6684	1	1	radial	
?	4.6922	?	1	radial	
?	4.7158	?	1	radial	
5	5.0000	2	2	shell function	$2 = 1 + 1$: Coming from two different shells.
9	9.0000	4	4	shell function	All supported on the same shell.
?	40.5213	?	2	radial	
$40+5/9 = 40.5555\ldots$	40.5555	2	2	shell function	$2 = 1 + 1$: Coming from two different shells.
41	41.0000	8	8	shell function	$8 = 4 + 4$: Coming from two different shells.
45	45.0000	24	24	shell function	$24 = 12 + 12$: Coming from two different shells.

Table 2. Eigenfunctions for three different eigenvalues, 28 values for each function, coming from all the 5 shells. Both kinds of eigenfunctions occur (shell functions and radial functions). – Shell no. k ($k = 2, 1, 0, -1, -\infty$) is the shell $|x| = 3^k$ (so shell no. $-\infty$ is the shell $|x| = 3^{-\infty} = 0$).

Eigenfunction for the lowest eigenvalue $\boxed{\lambda \approx 0.6684}$. It exhibits a perfect radial behavior. Notice also that the function is strictly positive, in accordance with the corresponding statement for the case $K = \mathbf{Q}_p$ in [VVZ94, p. 186].		Eigenfunction for $\boxed{\lambda = 5}$. Eigenfunctions here are linear combinations of shell functions from two different shells (shells 1 and 0). As should be expected, the function below exhibits non-radial behavior, being non-constant on each shell where it doesn't vanish (shells 1 and 0).		Eigenfunction for $\boxed{\lambda = 9}$. It exhibits a perfect shell function behavior, with support on shell no. 1.	
	Shell no.		Shell no.		Shell no.
$3.5818432 \cdot 10^{-1}$	$-\infty$	$1.8757870 \cdot 10^{-15} \approx 0$	$-\infty$	$-3.8765003 \cdot 10^{-16} \approx 0$	$-\infty$
$5.5430722 \cdot 10^{-5}$	2	$2.0896995 \cdot 10^{-16} \approx 0$	2	$1.6021680 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$8.7737711 \cdot 10^{-17} \approx 0$	2	$-9.1411700 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$-1.4801152 \cdot 10^{-16} \approx 0$	2	$5.1268297 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$3.0773313 \cdot 10^{-16} \approx 0$	2	$2.7677667 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$-4.5409159 \cdot 10^{-17} \approx 0$	2	$-4.5822760 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$-1.0479409 \cdot 10^{-16} \approx 0$	2	$-1.3758518 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$-2.3471948 \cdot 10^{-17} \approx 0$	2	$2.1385872 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$7.9466194 \cdot 10^{-17} \approx 0$	2	$-1.0549816 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$2.3950293 \cdot 10^{-16} \approx 0$	2	$2.3917324 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$6.4773691 \cdot 10^{-17} \approx 0$	2	$1.2912546 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$-1.1431061 \cdot 10^{-16} \approx 0$	2	$-6.0210598 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$-1.3177515 \cdot 10^{-17} \approx 0$	2	$-3.9251100 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$1.3595786 \cdot 10^{-16} \approx 0$	2	$-5.0103544 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$3.2839452 \cdot 10^{-17} \approx 0$	2	$1.2137971 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$7.8206625 \cdot 10^{-17} \approx 0$	2	$-1.0063910 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$3.3933100 \cdot 10^{-17} \approx 0$	2	$-7.7900493 \cdot 10^{-17} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$8.8459742 \cdot 10^{-17} \approx 0$	2	$2.2672330 \cdot 10^{-16} \approx 0$	2
$5.5430722 \cdot 10^{-5}$	2	$2.2115193 \cdot 10^{-17} \approx 0$	2	$-1.1819127 \cdot 10^{-16} \approx 0$	2
$1.2747433 \cdot 10^{-2}$	1	$-2.3459638 \cdot 10^{-1}$	1	$5.9907185 \cdot 10^{-2}$	1
$1.2747433 \cdot 10^{-2}$	1	$2.3459638 \cdot 10^{-1}$	1	$-4.1084268 \cdot 10^{-1}$	1
$1.2747433 \cdot 10^{-2}$	1	$-2.3459638 \cdot 10^{-1}$	1	$-1.0595734 \cdot 10^{-1}$	1
$1.2747433 \cdot 10^{-2}$	1	$2.3459638 \cdot 10^{-1}$	1	$2.7644342 \cdot 10^{-2}$	1
$1.2747433 \cdot 10^{-2}$	1	$-2.3459638 \cdot 10^{-1}$	1	$4.6050157 \cdot 10^{-2}$	1
$1.2747433 \cdot 10^{-2}$	1	$2.3459638 \cdot 10^{-1}$	1	$3.8319834 \cdot 10^{-1}$	1
$3.1960943 \cdot 10^{-1}$	0	$3.9500330 \cdot 10^{-2}$	0	$1.2637350 \cdot 10^{-17} \approx 0$	0
$3.1960943 \cdot 10^{-1}$	0	$-3.9500330 \cdot 10^{-2}$	0	$-1.6035100 \cdot 10^{-17} \approx 0$	0
$3.5768544 \cdot 10^{-1}$	-1	$2.2996138 \cdot 10^{-17} \approx 0$	-1	$-9.9411507 \cdot 10^{-17} \approx 0$	-1

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