On the de Sitter tardyons and tachyons

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Abstract

It is shown that on the de Sitter manifolds the tachyonic geodesics are restricted such that the classical tachyons cannot exist on this manifold at any time. On the contrary, the theory of the scalar quantum tachyons is free of any restriction

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Contents

Introduction

In special relativity one knows three types of particles, tardyons, light-like and tachyons. The first two types of particles are of our world, inside the light-cone, while the tachyons seems to live in another one, outside the light-cone. These two worlds seems to be completely separated as long as there are no direct physical evidences about the tardyon-tachyon interactions.

For this reason the tachyons are the most attractive hypothetical objects for speculating in some domain in physics where we have serious difficulties in building coherent theories. We mention, as an example, the presumed role of the tachyons in the early brane cosmology [1].

However, here we do not intend to comment on this topics, restricting ourselves to analyze, from the mathematical point of view, the possibility to meet classical or quantum scalar tachyons on the de Sitter backgrounds.

The de Sitter manifold, denoted from now by M, is local-Minkowskian such that the tachyons can be defined as in special relativity, with the difference that their properties are arising now from the specific high symmetry of the de Sitter manifolds.

Many authors exploited this high symmetry for building theories of quantum fields, either by constructing symmetric two-point functions, avoiding thus the canonical quantization [3], or by using directly these unitary representations for finding field equations but without considering covariant representations [4, 5].

Another approach which applies the canonical quantization to the covariant fields transforming according to induced covariant representations was initiated by Nachtmann [6] many years ago and continued in few of our papers [7, 8, 9].

In what follows we would like to study the tachyons on M in this last framework by using the traditional definition of tachyons as particles of real squared masses but of opposite sigs with respect to the tadyonic ones.

For this reason our results are different from other approaches [4, 5] where particles whose squared masses are supposed to be complex numbers are considered as tachyon [5].

Thus we find a result that seems to be somewhat paradoxical, namely that the classical tachyons cannot exist on M at any time while the quantum scalar particles behaves normally on this manifold, without any restriction.

Classical de Sitter geodesics

The de Sitter spacetime M is defined as the hyperboloid of radius $1/\omega$ in the five-dimensional flat spacetime (M^5, η^5) of coordinates z^A (labeled by the indices $A,\ B, ... = 0, 1, 2, 3, 4$) and metric $\eta^5 = \mathrm{diag}(1, -1, -1, -1, -1)$. The local charts $\{x\}$ can be introduced on (M, g) giving the set of functions $z^A(x)$ which solve the hyperboloid equation,

$$\eta^5_{AB} z^A(x) z^B(x) = -\frac{1}{\omega^2}\,. \tag{1}$$

Here we use the chart $\{t, \vec{x}\}$ with the conformal time t and Cartesian spaces coordinates x^i defined by

$$\begin{aligned} z^0(x) &= -\frac{1}{2\omega^2 t}\left[1 - \omega^2(t^2 - \vec{x}^2)\right] \\ z^i(x) &= -\frac{1}{\omega t} x^i\,, \\ z^4(x) &= -\frac{1}{2\omega^2 t}\left[1 + \omega^2(t^2 - \vec{x}^2)\right] \end{aligned} \tag{2}$$

This chart covers the expanding portion, M_+, of M for $t \in (-\infty, 0)$ and $\vec{x} \in \mathbb{R}^3$ while the collapsing part, M_-, is covered by a similar chart but with $t > 0$. Both these charts have the conformal flat line element,

$$ds^2 = \eta^5_{AB} dz^A(x) dz^B(x) = \frac{1}{\omega^2 t^2}\left(dt^2 - d\vec{x}^2\right) \, . \tag{3}$$

We remind the reader that the proper time is defined on each portion in the domain $(-\infty, \infty)$ as,

$$t_{proper} = \begin{cases} -\frac{1}{\omega}\ln(-\omega t) \;\text{ on }\; M_+ \;\text{ for }\; -\infty < t < 0\,, \\ \\ \;\frac{1}{\omega}\ln(\omega t) \quad\; \text{ on }\; M_- \;\text{ for }\; 0 < t < \infty\,. \end{cases} \tag{4}$$

By definition, the de Sitter spacetime M is a homogeneous space of the pseudo-orthogonal group $SO(1,4)$ which is in the same time the gauge group of the metric η^5 and the isometry group, $I(M)$, of M.

The classical conserved quantities as well as the basis-generators of the covariant representations of the isometry group can be calculated with the help of the Killing vectors $k_{(AB)}$ which have the following components:

$$k_{(0i)}^{0} = k_{(4i)}^{0} = \omega t x^{i}\,, \qquad k_{(0i)}^{j} = k_{(4i)}^{j} - \frac{1}{\omega}\delta_{i}^{j} = \omega x^{i} x^{j} - \delta_{i}^{j}\chi\,, \tag{5}$$

$$k_{(ij)}^{0} = 0\,, \quad k_{(ij)}^{l} = \delta_{j}^{l} x^{i} - \delta_{i}^{l} x^{j}\,; \qquad k_{(04)}^{0} = t\,, \quad k_{(04)}^{i} = x^{i}\,. \tag{6}$$

where

$$\chi = \frac{1}{2\omega}\left[1 - \omega^{2}(t^{2} - \vec{x}^{2})\right]\,. \tag{7}$$

Furthermore, It is a simple exercise to integrate the geodesic equations and to find the conserved quantities on a geodesic trajectory on M.

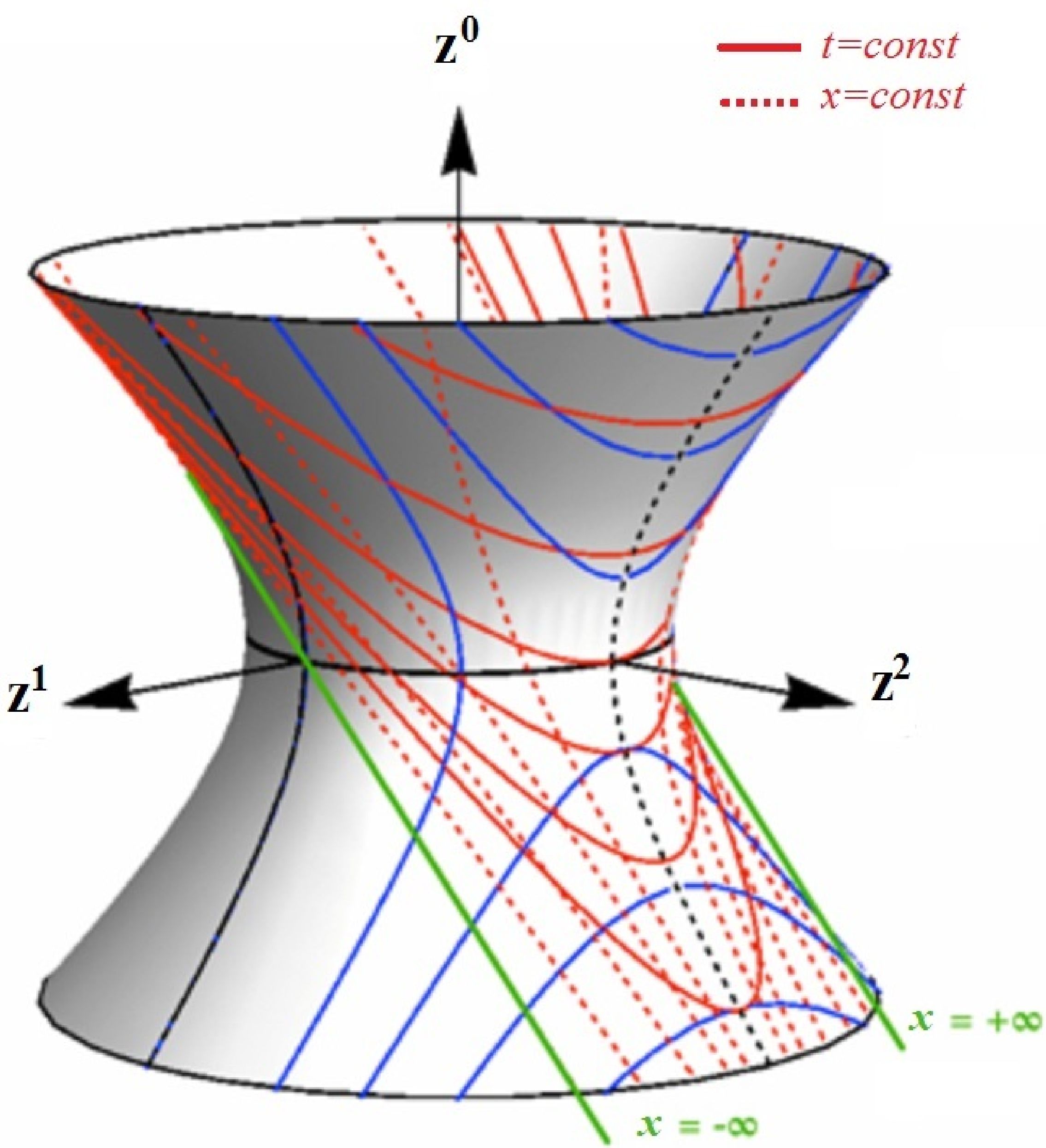

Figure 1: The expanding portion of the de Sitter hyperboloid in the case of $d = 1$

Using the standard notation $u^\mu = \frac{dx^\mu(s)}{ds}$ we bear in mind that the principal invariant $u^2 = g_{\mu\nu}u^\mu u^\nu = \epsilon$ along the geodesics $x = x(s)$ determines the type of this trajectory, i. e. tardyonic ($\epsilon = 1$), light-like ($\epsilon = 0$) or tachyonic ($\epsilon = -1$). All the other conserved quantities along the geodesics are proportional to $k_{(AB)\,\mu}u^\mu$ and can be derived by using the Killing vectors (5) and (6).

We assume first that in the chart $\{t, \vec{x}\}$ the particle of mass $m \neq 0$ has the conserved momentum $\vec{p}$ of components $p^{\,i} = \omega m(k_{(0i)\,\mu} - k_{(4i)\,\mu})u^\mu$ so that we can write

$$u^0 = \frac{dt}{ds} = -\omega t\sqrt{\epsilon + \frac{\omega^2 p^2}{m^2}t^2}\,, \qquad u^i = \frac{dx^i}{ds} = -\omega t\frac{p^{\,i}}{m}\,, \tag{8}$$

using the notation $p = |\vec{p}\,|$. Hereby we deduce the trajectory,

$$x^i(t) = x_0^i + \frac{p^{\,i}}{\omega p^2}\left(\sqrt{\epsilon m^2 + p^2\omega^2 t_0^2} - \sqrt{\epsilon m^2 + p^2\omega^2 t^2}\right)\,, \tag{9}$$

of a massive particle passing through the point $\vec{x}_0$ at time t_0. The light-like case must be treated separately finding that the trajectory of a massless particle reads

$$x^i(t) = x_0^i + n^i\,(t_0 - t)\ , \tag{10}$$

where the unit vector $\vec{n}$ gives the propagation direction.

Among the conserved quantities that can be derived as in Ref. [8] the energy of the massive particles,

$$E = \omega\,\vec{x}_0 \cdot \vec{p} + \sqrt{\epsilon m^2 + p^2\omega^2 t_0^2}\,, \tag{11}$$

indicates the allowed domains of our parameters. Thus we see that the tardyonic particles can have any momentum, $\vec{p} \in \mathbb{R}^3_p$, since their energies remain always real numbers. When the tardyonic particle is at rest, staying in $\vec{x}(t) = \vec{x}_0$ with $\vec{p} = 0$, then we find the same rest energy $E_0 = m$ as in special relativity. Thus we conclude that the tardyonic particles behave familiarly just as in Minkowski flat spacetime.

However, the tachyons have a new and somewhat strange behavior. When $\epsilon = -1$ there are real trajectories only for large momentum with $|\vec{p}| > m$, corresponding to the super-luminary motion. Moreover, we must have $p^2\omega^2 t_0^2 \geq m^2$ which means that t_0 satisfies either the condition $t_0 \leq -\frac{m}{p\omega}$ on M_+ or the symmetric one $t_0 \geq \frac{m}{p\omega}$ on M_-.

Thus we conclude that a classical tachyon of momentum $p > m$ can exist on the expanding portion only in the time domain $-\infty < t \leq -\frac{m}{p\omega}$ while on the collapsing portion its time domain is $\frac{m}{p\omega} \leq t < \infty$.

These restrictions upon the tachyonic lifetime are quite unusual opening the problem of finding a plausible mechanism explaining the tachyon death at the time $t = -\frac{m}{p\omega}$ on M_+ or how this is born at $t = \frac{m}{p\omega}$ on M_-.

Fortunately, for the quantum tachyons such restrictions seem to do not occur.

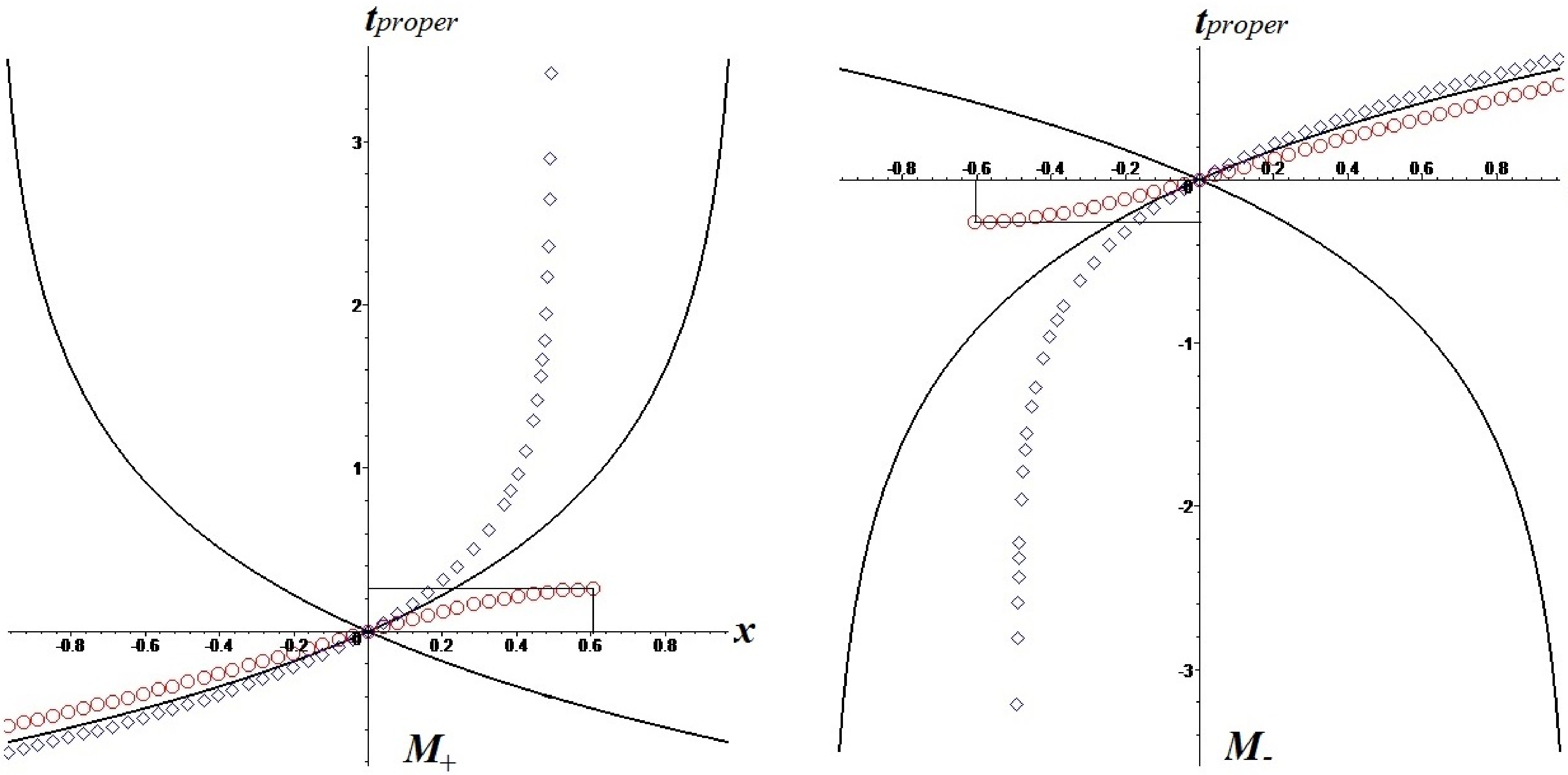

Figure 2: The worldlines of the tardyons (blue) and tachyons (red) with the same momentum and initial conditions $x_0 = 0$ at $t_0 = \mp\frac{1}{\omega}$ on M_+ (left panel) and M_- (right panel). The continuous lines represent the light-cones which tends asymptoticly to the event horizon.

Scalar plane waves

The specific feature of the quantum mechanics on M is that the energy operator does not commute with the components of the momentum operator. Therefore, the energy and momentum cannot be measured simultaneously with a desired accuracy.

Consequently, there are no particular solutions of the KG equation with well-determined energy and momentum, being forced to consider different plane waves solutions depending either on momentum or on energy and momentum direction. In what follows we restrict ourselves to the plane waves of determined momentum.

In an arbitrary chart $\{x\}$ the action of a charged scalar field ϕ of mass m, *minimally* coupled with the gravitational field, reads

$$\mathcal{S}[\phi,\phi^*]=\int d^4x\sqrt{g}\,\mathcal{L}=\int d^4x\sqrt{g}\left(\partial^\mu\phi^*\partial_\mu\phi-\epsilon m^2\phi^*\phi\right)\,, \tag{12}$$

where $g = |\det(g_{\mu\nu})|$ and ϵ is our parameter which gives the tardyonic, tachyonic or light-like behaviour.

This action gives rise to the KG equation

$$\frac{1}{\sqrt{g}}\,\partial_\mu\left[\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi\right] + \epsilon m^2\phi = 0\,, \tag{13}$$

whose solutions have to be normalized (in generalized sense) with respect to the relativistic scalar product [10],

$$\langle\phi,\phi'\rangle = i\int_\Sigma d\sigma^\mu\sqrt{g}\,\phi^*\stackrel{\leftrightarrow}{\partial_\mu}\phi'\,, \tag{14}$$

written with the notation $f\stackrel{\leftrightarrow}{\partial}h = f(\partial g) - g(\partial f)$.

In the chart $\{t,\vec{x}\}$ we use here the Klein-Gordon equation takes the form,

$$\omega^2 t^2\left(\partial_t^2 - \frac{2}{t}\partial_t - \Delta\right)\phi(x) + \epsilon m^2\psi(x) = 0\,. \tag{15}$$

The solutions of this equation may be square integrable functions or tempered distributions with respect to the scalar product (14) that for $\Sigma = \mathbb{R}^3$ takes the form

$$\langle \phi, \phi' \rangle = i \int d^3x \, e^{3\omega t} \, \phi^*(x) \overleftrightarrow{\partial_t} \, \phi'(x) \, . \tag{16}$$

It is known that the KG equation (15) of NP can be analytically solved in terms of Bessel functions [10]. There are fundamental solutions of positive frequencies and given momentum, $\vec{p}$, that read

$$f_{\vec{p}}(x) = \frac{1}{2}\sqrt{\frac{\pi}{\omega}} \frac{e^{-\frac{1}{2}i\pi k}}{(2\pi)^{3/2}} (-\omega t)^{\frac{3}{2}} H_k^{(1)}(-pt) \, e^{i\vec{p}\cdot\vec{x}} \, , \tag{17}$$

where $H_\nu^{(1)}$ are Hankel functions, $p = |\vec{p}|$ and we denote

$$k = \sqrt{\frac{9}{4} - \epsilon \frac{m^2}{\omega^2}} \, . \tag{18}$$

Obviously, the fundamental solutions of negative frequencies are $f^*_{\mathbf{p}}(x)$.

Now we observe that the only parameter depending on ϵ is just that given by Eq. (18) which encapsulates the information about the nature of the scalar particle. We see first that there are no major differences between the tardyonic ($\epsilon = 1$) and tachyonic ($\epsilon = -1$) cases.

The only property depending on the particle's nature is the behavior of the mode functions in the limit of $t \to 0$, when the proper time tends to ∞ on M_+ or to $-\infty$ on M_- as in Eq. (4). The functions (17) are finite in the considered limit only if we have $k \leq \frac{3}{2}$ which holds only for $\epsilon = 1$. Thus we find that the tardyonic mode functions remain finite for $t \to 0$ while the tachyonic ones diverge in this limit.

However, this is not an impediment as long as the conserved scalar product and the conserved quantities do not depend explicitly on the functions giving the time modulation.

The conclusion here is that the quantum scalar tachyons could live on M at any time.

Dirac plane waves

The tardyonic free Dirac field ψ of mass m and minimally coupled to the gravity of M has the action

$$\mathcal{S}[\psi] = \int d^4x\sqrt{g}\,\left(\mathcal{L}_D(\psi) - m\overline{\psi}\psi\right)\,. \tag{19}$$

where the massless Lagrangian density, [11],

$$\mathcal{L}_D(\psi) = \frac{i}{2}\,[\overline{\psi}\gamma^{\hat\alpha}D_{\hat\alpha}\psi - (\overline{D_{\hat\alpha}\psi})\gamma^{\hat\alpha}\psi]\,, \quad \overline{\psi} = \psi^+\gamma^0\,, \tag{20}$$

depends on the covariant derivatives in local frames, $D_{\hat\alpha}$ [11], that guarantee the tetrad-gauge invariance.

The point-independent Dirac matrices $\gamma^{\hat\mu}$ satisfy $\{\gamma^{\hat\alpha},\,\gamma^{\hat\beta}\} = 2\eta^{\hat\alpha\hat\beta}$ giving rise to the basis-generators $S^{\hat\alpha\hat\beta} = i[\gamma^{\hat\alpha},\gamma^{\hat\beta}]/4$ of the spinor representation $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ of the $SL(2,\mathbb{C})$ group that induces the spinor covariant representations [11].

The Lagrangian theory of the Dirac tachyons [12, 13, 14] allows us to introduce our parameter ϵ for studying simultaneously the tardyonic, neutrino and tachyonic cases. This can be done on a natural way considering the new action

$$\mathcal{S}_\epsilon[\psi_L,\psi_R]=\int d^4x\sqrt{g}\,\left[\mathcal{L}_D(\psi_L)+\epsilon\mathcal{L}_D(\psi_R)-\epsilon m(\overline{\psi}_L\psi_R+\overline{\psi}_R\psi_L)\right]\,, \tag{21}$$

depending on the chiral projections $\psi_L=L\psi$ and $\psi_R=R\psi$ given by the standard projectors

$$L=\frac{1-\gamma^5}{2}\,,\quad R=\frac{1+\gamma^5}{2}\,. \tag{22}$$

Note that this action is written in the style of the Standard Model such that the left-handed term is independent on ϵ in order to do not affect the $SU(2)_L$ gauge symmetry.

The action (21) gives the field equations of the massive fields,

$$\begin{aligned} \epsilon = 1 \quad & (i\gamma^{\hat{\alpha}} D_{\hat{\alpha}} - m)\psi_{tard} = 0 \quad \text{tardyon} \\ \epsilon = -1 \ & (i\gamma^5\gamma^{\hat{\alpha}} D_{\hat{\alpha}} + m)\psi_{tach} = 0 \ \text{tachyon} \end{aligned} \tag{23}$$

while for $\epsilon = 0$ we recover the usual theory of the massless neutrino. It is remarkable that these equations are related through

$$\psi_{tach}(x, m) = \tau\psi_{tard}(x, im) \tag{24}$$

where the unitary matrix

$$\tau = \frac{1}{\sqrt{2}}(1 - i\gamma^5) \tag{25}$$

has the properties $\tau^+ = \tau^{-1}$, $\overline{\tau} = \tau$ and $\tau^2 = -i\gamma^5$. This means that it is enough to solve the tardyonic equation for finding simultaneously the tachyonic solution by using Eq. (24)

Let us start with the tardyonic case in the chart $\{t,\vec{x}\}$ and tetrad-gauge

$$e^0_0=-\omega t\,,\quad e^i_j=-\delta^i_j\,\omega t\,,\quad \hat{e}^0_0=-\frac{1}{\omega t}\,,\quad \hat{e}^i_j=-\delta^i_j\,\frac{1}{\omega t}\,. \tag{26}$$

where the free Dirac equation for tardyons reads [11],

$$\left[-i\omega t\left(\gamma^0\partial_t+\gamma^i\partial_i\right)+\frac{3i\omega}{2}\gamma^0-m\right]\psi(x)=0\,. \tag{27}$$

The general solution allows the mode expansion in momentum representation,

$$\psi(t,\vec{x})=\int d^3p\sum_{\sigma}\left[U_{\vec{p},\sigma}(x)a(\vec{p},\sigma)+V_{\vec{p},\sigma}(x)a^{c\dagger}(\vec{p},\sigma)\right]\,, \tag{28}$$

written in terms of the field operators, a and a^c, and the particle and antiparticle fundamental spinors of this basis, $U_{\vec{p},\sigma}$ and respectively $V_{\vec{p},\sigma}$, which depend on the momentum $\vec{p}$ (with $p=|\vec{p}|$) and polarization $\sigma=\pm 1/2$.

According to our prescription of frequencies separation on the expanding portion (of the Bounch-Davies type) we find that these spinors, in the standard representation of the Dirac matrices (with diagonal γ^0), take the form [11],

$$U_{\vec{p},\sigma}(t,\vec{x}\,) = iN(\omega t)^2 \left(\begin{array}{c} e^{\frac{1}{2}\pi\mu} H^{(1)}_{\nu_-}(-pt)\,\xi_\sigma \\ e^{-\frac{1}{2}\pi\mu} H^{(1)}_{\nu_+}(-pt)\,\frac{\vec{\sigma}\cdot\vec{p}}{p}\,\xi_\sigma \end{array} \right) e^{i\vec{p}\cdot\vec{x}} \tag{29}$$

$$V_{\vec{p},\sigma}(t,\vec{x}\,) = -iN(\omega t)^2 \left(\begin{array}{c} e^{-\frac{1}{2}\pi\mu} H^{(2)}_{\nu_-}(-pt)\,\frac{\vec{\sigma}\cdot\vec{p}}{p}\,\eta_\sigma \\ e^{\frac{1}{2}\pi\mu} H^{(2)}_{\nu_+}(-pt)\,\eta_\sigma \end{array} \right) e^{-i\vec{p}\cdot\vec{x}}\,. \tag{30}$$

The notation σ_i stands for the Pauli matrices while $H^{(1,2)}_{\nu_\pm}$ are the Hankel functions of indices $\nu_\pm = \frac{1}{2} \pm i\mu$ with $\mu = \frac{m}{\omega}$. The normalization constant N has to be calculated according to a normalization condition on M_+ that will not be discussed here. Similar solutions can be obtained on M_- by changing $\omega \to -\omega$.

Now we can write directly the tachyonic fundamental solutions using Eq. (24). We obtain the final result as

$$\tilde{U}_{\vec{p},\sigma}(t,\vec{x}\,) = i\tilde{N}(\omega t)^2 \left[\begin{array}{c} \left(e^{\frac{i}{2}\pi\mu} H^{(1)}_{\tilde{\nu}_+}(-pt) - i e^{-\frac{i}{2}\pi\mu} H^{(1)}_{\tilde{\nu}_-}(-pt)\, \frac{\vec{\sigma}\cdot\vec{p}}{p} \right) \xi_\sigma \\ \left(e^{-\frac{i}{2}\pi\mu} H^{(1)}_{\tilde{\nu}_-}(-pt)\, \frac{\vec{\sigma}\cdot\vec{p}}{p} - i e^{\frac{i}{2}\pi\mu} H^{(1)}_{\tilde{\nu}_+}(-pt) \right) \xi_\sigma \end{array} \right] e^{i\vec{p}\cdot\vec{x}}\,, \quad (31)$$

$$\tilde{V}_{\vec{p},\sigma}(t,\vec{x}\,) = -i\tilde{N}(\omega t)^2 \left[\begin{array}{c} \left(e^{-\frac{i}{2}\pi\mu} H^{(2)}_{\tilde{\nu}_+}(-pt)\, \frac{\vec{\sigma}\cdot\vec{p}}{p} - i e^{\frac{i}{2}\pi\mu} H^{(2)}_{\tilde{\nu}_-}(-pt) \right) \eta_\sigma \\ \left(e^{\frac{i}{2}\pi\mu} H^{(2)}_{\tilde{\nu}_-}(-pt) - i e^{-\frac{i}{2}\pi\mu} H^{(2)}_{\tilde{\nu}_+}(-pt)\, \frac{\vec{\sigma}\cdot\vec{p}}{p} \right) \eta_\sigma \end{array} \right] e^{-i\vec{p}\cdot\vec{x}}\,, \quad (32)$$

where now all the indices $\tilde{\nu}_\pm = \frac{1}{2} \pm \frac{m}{\omega}$ (with $\mu = \frac{m}{\omega}$) are real numbers. These fundamental spinors are defined on the whole expanding portion without restrictions despite of their complicated form. On the collapsing portion we meet similar solutions but with $\omega \to -\omega$ and $t > 0$.

We note that, as in the scalar case, there are mode functions that diverge in the limit of $t \to 0$ but only for the tachyonic mass $m > \frac{3}{2}\omega$ while for $m < \frac{3}{2}\omega$ these functions remain finite as in the tardyonic case. This situation is different from the scalar case where all the tachyonic scalar functions are divergent in this limit. However, this phenomenon has no physical consequences since only the conserved quantities (scalar products and expectation values) have physical significance.

Thus we can say that the Dirac tachyons on the de Sitter background have fundamental solutions well-defined on the entire physical space at any moment.

Conclusion

Finally, we must stress that the tachyon physics on the de Sitter manifold seems to lead to a fundamental contradiction between the classical and quantum cases. Thus in the classical approach the trajectory must end when the energy becomes imaginary. On the contrary, in the quantum theory there are no time restrictions for the mode functions that behave as tempered distributions on the whole physical space.

The present paper is unable to solve this discrepancy since here we present the tachyonic solutions without studying the conserved quantities which could offer new arguments in what concerns the relation between the classical and quantum tachyons. However, in this domain one could face to serious difficulties but that could be overdrawn by using new analytical and even numerical methods [15].

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