Minimal Liouville Gravity from Douglas string equation

A.B.

(Based on joint works with V.Belavin, B. Dubrovin, B. Mukhametzhanov, A.Zamolodchikov,)

Landau Institute for theoretical physics

Introduction

We use the connection between the Frobrenius manifold and Douglas string equation for the further investigation Minimal Liouville gravity.

The Liouville Gravity is a synonym of the noncritical String theory. LG is formulated as a BRST invariant CFT ,tensor of the matter sector, the Liouville theory and the ghosts system. In Minimal Liouville Gravity (MLG) the matter sector is a (p, q) Minimal CFT Model .

An alternative approach to 2d gravity has grown up from the idea of triangulations of two-dimensional surfaces realized as Matrix models.

The main tool of this approach is String equation which was derived by Douglas in framework of Matrix models of 2d gravity..

The subject of the String equation is the generating function of the correlation numbers which depends on parameters of the problem (KdV times).

In our work we conjecture that Douglas equation is applicable to the Minimal Liouville gravity as well as to Matrix models of two dimensional gravity.

We try to solve the problem how to choose and is it possible to choose the appropriate solution of the Douglas string equation and the appropriate transformation from KdV frame to Liouville frame to ensure the fulfilement the fusion rules of the MLG .

Using the connection of the Douglas approach with the Frobenius manifold structure we find the necessary solution of String equation and argue that this solution together with the resonance transformation do exist and lead to the correct results .

The Minimal Model $\mathcal{M}_{q,p}$ has primary fields, which correspond to integrable representations of Virasoro algebra and which are enumerated by elements of the Kac table: $\Phi_{m,n}$, where $m = 1, \ldots, q-1$ and $n = 1, \ldots, p-1$. Only half of the fields $\Phi_{m,n}$ are independent

$$\Phi_{m,n} = \Phi_{q-m,p-n}.$$

The operator product expansion (OPE) for these fields is the subject of the following fusion rules

$$[\Phi_{m_1,n_1}][\Phi_{m_2,n_2}] = \sum_{m=|m_1-m_2|:2}^{I(m_1,m_2)} \sum_{n=|n_1-n_2|:2}^{I(n_1,n_2)} [\Phi_{m,n}],$$

where $[\Phi_{m,n}]$ denotes the contribution of the irreducible Virasoro representation with the highest state $\Phi_{m,n}$. The summation goes with the step 2 and

$$I(a,b) = \min(a+b-1, 2q-a-b-1).$$

The small conformal group and OPE give strong constraints on the correlation functions. The constraints for one- and two- point correlation functions are

$$\langle \Phi_{m,n}(x) \rangle = 0,$$

$$\langle \Phi_{m_1,n_1}(x_1) \Phi_{m_2,n_2}(x_2) \rangle = 0, \qquad m_1, n_1 \neq m_2, n_2.$$

For higher correlation numbers we also find some restriction which follow from the OPE fusion rules. For instance the three-point correlation functions satisfy

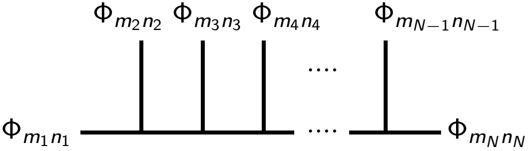
$$\langle \Phi_{m_1,n_1} \Phi_{m_2,n_2} \Phi_{m_3,n_3} \rangle = 0,$$

for

$$m_3 > I(m_1, m_2) = \begin{cases} m_1 + m_2 - 1, & m_1 + m_2 - 1 \leq q - 1 \\ 2q - m_1 - m_2 - 1, & m_1 + m_2 > q \end{cases},$$

where we assume that $m_1 \leq m_2 \leq m_3$.

The following graphical representation allows simply to formulate these restrictions for the general correlation function



Here the external lines represent the primary fields in the correlator $\langle \Phi_{m_1 n_1} \Phi_{m_2 n_2} ... \Phi_{m_N n_N} \rangle$.

From the fusion rules it follows that the correlator has to be zero if we cannot assign some set of pairs (k_i, l_i) to the internal lines, in such a way that in each vertex of the graph the following condition for the three pairs corresponding to the lines connected to this vertex is fulfilled

$$|m_1 - m_2| + 1 \leq m_3 \leq \min\{m_1 + m_2 - 1, 2q + 1 - m_1 - m_2\},$$
$$|n_1 - n_2| + 1 \leq n_3 \leq \min\{n_1 + n_2 - 1, 2p + 1 - n_1 - n_2\},$$

for any permutation of the pairs. These equations represent so called selection rules.

Liouville Field Theory

The Polyakov's continuous approach to two-dimensional quantum gravity is defined through the path integral over two-dimensional Riemannian metrics $g_{\mu\nu}$ interacting with some conformal matter. Because of the conformal anomaly, in the conformal gauge $g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}$ it leads to the Liouville action

$$S_L = \frac{1}{4\pi}\int_M \sqrt{\hat{g}}\left(\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + Q\hat{R}\phi + 4\pi\mu e^{2b\phi}\right) d^2x,$$

where $\hat{g}_{\mu\nu}$ is some fixed background metric, μ is the cosmological constant and parameters Q, b are related to the central charge c_L of the Liouville theory

$$c_L = 1 + 6Q^2, \qquad Q = b + b^{-1}.$$

The central charge c_M of the conformal matter is related to the central charge of the Liouville theory by the requirement of the cancellation of the Weyl anomaly

$$c_L + c_M = 26.$$

In the case of (q, p) Minimal Liouville Gravity, where the conformal matter is (q, p) Minimal Model of CFT, we find $b = \sqrt{\frac{q}{p}}$.

Correlation numbers of MLG

The observables of the (q,p) MLG are cohomologies of the BRST operator. They are in one-to one correspondence with the primary fields in the Minimal Model in the matter sector. We denote them $O_{m,n}$. Explicitly,

$$O_{m,n} = \int_{x\in M} \mathcal{O}_{m,n}(x), \qquad \mathcal{O}_{m,n}(x) = \Phi_{m,n}(x) e^{2b\delta_{m,n}\phi(x)} \sqrt{\hat{g}} d^2x.$$

The operators $O_{m,n}$ satisfy the same selection rules as $\Phi_{m,n}$. Moreover, they have the following scaling property

$$O_{m,n} \sim \mu^{-\delta_{m,n}}, \qquad \delta_{m,n} = \frac{p+q-|pm-qn|}{2q}.$$

The correlation numbers in Minimal Liouville Gravity are defined as

$$Z_{m_1 n_1 \ldots m_N n_N} = \langle O_{m_1,n_1} \ldots O_{m_N,n_N} \rangle.$$

The generating function of these correlation numbers is

$$Z_L(\{\lambda_{m,n}\}) = \left\langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \right\rangle .$$

It is a quasihomogeneous function, i.e.,

$$Z_L(\{\rho^{\delta_{m,n}} \lambda_{m,n}\}) = \rho^{\frac{p+q}{q}} Z_L(\{\lambda_{m,n}\}).$$

Contact terms

The correlation numbers involve integration over n points on the 2D surface X

$$Z_{m_1 n_1 \dots m_N n_N} = \int \langle \mathcal{O}_{m_1,n_1}(x_1) \dots \mathcal{O}_{m_N,n_N}(x_N) \rangle d^2 x_1 \dots d^2 x_{N-3}.$$

The contact delta-like terms may take place when two or more points x_i are coincident. The ambiguity in contact terms leads to the fact that we can add to the n-point correlation numbers some k-point correlation numbers

$$\langle O_{m_1,n_1} O_{m_2,n_2} \rangle \to \langle O_{m_1,n_1} O_{m_2,n_2} \rangle + \sum_{m,n} A_{m,n}^{(m_1 n_1)(m_2 n_2)} \langle O_{m,n} \rangle.$$

This substitution is equivalent to the change of coupling constants in the generating function

$$\lambda_{m,n} \to \lambda_{m,n} + \sum_{m_1,n_1,m_2,n_2} A_{m,n}^{(m_1 n_1)(m_2 n_2)} \lambda_{m_1 n_1} \lambda_{m_2 n_2}.$$

In MLG there is some restrictions on this change of coupling constants since they have certain scaling dimension

$$\lambda_{m,n} \sim \mu^{\delta_{m,n}}.$$

Therefore we can demand that all the terms be of the same dimensions

$$\delta_{m,n} = \delta_{m_1,n_1} + \delta_{m_2,n_2} + \delta_{m_3,n_3} + \ldots.$$

The addition of the contact terms is equivalent to some non-linear (polynomial) change of the coupling constants

$$\lambda_{m,n} \to A\mu^{\delta_{m,n}} + \sum_{m_1,n_1} C_{m,n}^{(m_1 n_1)} \mu^{\delta_{m,n}-\delta_{m_1,n_1}} \lambda_{m_1,n_1} \tag{0.1}$$

$$+ \sum_{m_1,n_1} \sum_{m_2,n_2} C_{m,n}^{(m_1 n_1)(m_2 n_2)} \mu^{\delta_{m,n}-\delta_{m_1,n_1}-\delta_{m_2,n_2}} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \ldots \tag{0.2}$$

Only the terms in these sums with the integer and positive degrees

$$\delta_{m,n} - \delta_{m_1,n_1} - \delta_{m_2,n_2} - \ldots .$$

have nonvanishing coefficients. So, there exist different “systems of coordinates"of $\lambda_{m,n}$ and, in general case, the MLG coordinate frame does not coincide with the natural coordinate system for Douglas string equation. The change of variables conserves the quasi-homogeneity property

$$Z_L(\{\rho^{\delta_{m,n}}\lambda_{m,n}\}) = \rho^{\frac{p+q}{q}} Z_L(\{\lambda_{m,n}\}).$$

Frobenius manifolds

A commutative and associative algebra A with unity, equipped with a nondegenerate invariant pairing $(\,,\,)$ is called Frobenius algebra. The invariance means that for any three vectors a, b, c in A:

$$(a \cdot b, c) = (a, b \cdot c).$$

Let M be n-dimensional manifold with flat metric $\eta_{\alpha\beta} dv^\alpha dv^\alpha$ which is constant in the flat coordinates v^α.
We introduce in the tangent space $T_{\mathbf{v}}M$ the structure of the Frobenius algebra by means of the following identification of the bases

$$\frac{\partial}{\partial v^\alpha} \to e_\alpha,$$

Thus, we can multiply tangent vectors at any point of M

$$e_\alpha e_\beta = C^\gamma_{\alpha\beta} e_\gamma.$$

The structure constants $C^{\gamma}_{\alpha\beta}$ may depend on v^{α}. Such manifold M can be called quasi-Frobenius manifold.

Definition: The manifold M is called Frobenius manifold if these two structures are adjusted with each other in such a way that the bilinear form $(\frac{\partial}{\partial v^{\alpha}}, \frac{\partial}{\partial v^{\beta}})$ of Frobenius algebra is identical to the metric $\eta_{\alpha\beta}$;
the structure of the Frobenius algebra in each point of M and the metric on M are connected by the following relation

$$\nabla_{\rho} C_{\alpha\beta\gamma} = \nabla_{\alpha} C_{\rho\beta\gamma}.$$

This is equivalent to the requirement that there exists a function F on M which is connected with the structure constants of the Frobenius algebra as

$$C_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^{\alpha} \partial v^{\beta} \partial v^{\gamma}},$$

where

$$C_{\alpha\beta\gamma} = \eta_{\alpha\rho} C^{\rho}_{\beta\gamma}.$$

Function F is called Frobenius potential. The consistency of this property with the associativity of the Frobenius algebra is known as WDVV condition

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\rho} \eta^{\rho\lambda} \frac{\partial^3 F}{\partial v^\lambda \partial v^\mu \partial v^\nu} = \frac{\partial^3 F}{\partial v^\nu \partial v^\beta \partial v^\rho} \eta^{\rho\lambda} \frac{\partial^3 F}{\partial v^\lambda \partial v^\mu \partial v^\alpha}.$$

The following statement follows from these properties of the Frobenius manifold M. There exist one-parametric flat deformation $\widetilde{\nabla}_\alpha$ of the connection ∇_α

$$\widetilde{\nabla}_\alpha V^\gamma = \nabla_\alpha V^\gamma - z C^\gamma_{\alpha\beta} V^\beta,$$

or, equivalently,

$$[\widetilde{\nabla}_\alpha(z), \widetilde{\nabla}_\beta(z)] = 0.$$

The proof is based on the associativity of the Frobenius algebra and the WDVV equation .

It follows that there exist n linear independent solutions

$$\theta^\alpha(v,z)=\sum_{k=0}^{\infty}\theta_k^\alpha(v)z^k,$$

of the equation $\widetilde{\nabla}_\alpha d\theta^\lambda(v,z)=0$.
The last equation is equivalent to following recusive relations

$$\frac{\partial^2\theta_{k+1}^\lambda}{\partial v^\alpha\partial v^\beta}(v)=C_{\alpha\beta}^\gamma\frac{\partial\theta_k^\lambda}{\partial v^\gamma}(v).$$

The functions $\theta^\alpha(v,z)$ can be considered as the flat coordinates of the deformed connection $\widetilde{\nabla}_\alpha(z)$. We choose $\theta^\lambda(v,z)$ so that $\theta^\lambda(v,0)=\theta_0^\lambda(v)=v^\lambda$.

Frobenius manifold of A_{q-1} type

Let $Q(y)$ be a polynomial of y

$$Q(y) = y^q + u_1 y^{q-2} + ... + u_{q-1},$$

and $\{u_\alpha\}$ represent some coordinates on M. We call $\{u_i\}$ the canonical coordinates.
A_{q-1} Frobenius algebra is the space of polynomials of y modulo polynomial $\frac{dQ}{dy}$:

$$A_{q-1}(u) = \mathbb{C}[y]/\frac{dQ}{dy}.$$

The corresponding manifold M is called the Frobenius manifold of A_{q-1} type.
The canonical basis of A_{q-1}

$$P_i(y) = \frac{\partial Q}{\partial u_i},$$

An invariant bilinear form (which is equivalent to the metric)

$$(P_1, P_2) = \operatorname*{res}_{y=\infty}\left(\frac{P_1(y)P_2(y)}{\frac{dQ}{dy}(y)}\right).$$

Flat coordinates

With this definition one can verify that the corresponding metric is flat and

$$C_{\alpha\beta\gamma} = \nabla_\alpha \nabla_\beta \nabla_\gamma F(u).$$

To this end we perform the transformation from the canonical $\{u_\alpha\}$ to the new flat coordinates $\{v_\alpha\}$ by means of the following relation

$$y = z - \frac{1}{q}\left(\frac{v^{q-1}}{z} + \frac{v^{q-2}}{z^2} + \cdots + \frac{v^1}{z^{q-1}}\right) + \mathcal{O}\left(\frac{1}{z^{q+1}}\right),$$

where $z^q = Q(y)$.
In the flat coordinates

$$\eta^{\alpha\beta} = -q\left(\frac{\partial Q}{\partial v_\alpha}, \frac{\partial Q}{\partial v_\beta}\right) = \delta_{\alpha+\beta,q},$$

$$C_{\alpha\beta\gamma} = -q \operatorname*{res}_{y=\infty}\left(\frac{\frac{\partial Q}{\partial v^\alpha}\frac{\partial Q}{\partial v^\beta}\frac{\partial Q}{\partial v^\gamma}}{\frac{dQ}{dy}}\right) = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma},$$

The coefficients of the deformed flat coordinates $\theta_{\alpha,k}$ are expressed through the polynomial $Q(y)$ as

$$\theta_{\alpha,k} = -c_{\alpha,k} \operatorname*{res}_{y=\infty} Q^{k+\frac{\alpha}{q}}(y),$$

where

$$c_{\alpha,k} = \frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q}+k+1)}$$

Frobenius manifolds and string equation

We define a function $S(v,t)$ on M which depends on the additional parameters $\{t_k^\alpha\}$

$$S(v,t_k^\alpha)=\sum_{\alpha=1}^{n}\sum_{k\geq 0}t_k^\alpha\theta_{\alpha,k}(v).$$

The equation

$$\frac{\partial S}{\partial v^\alpha}=0,$$

is called a string equation. In the case of FM of A_{q-1} type it is just the Douglas string equation.

Define the function $Z[t]$

$$Z[t]=\frac{1}{2}\int_0^{v=v^*(t)}\Omega,$$

where

$$\Omega=C_\alpha^{\beta\gamma}(v)\frac{\partial S(v,t)}{\partial v^\beta}\frac{\partial S(v,t)}{\partial v^\gamma}dv^\alpha$$

and $v^*(t)$ is one of the solutions of the String equation .

From Jacobi and WDVV equations follows that Ω is closed one-form.

One can show that Z(t) satisfy

$$\frac{\partial^2 Z(t)}{\partial t_k^\alpha \partial t_0^1} = \theta_{\alpha,k}(v^*(t)).$$

In particular,

$$\frac{\partial^2 Z}{\partial t_0^\beta \partial t_0^1} = v_\beta^*(t),$$

and for $v_1(t) = u_1(t)$

$$\frac{\partial^2 Z}{\partial x^2} = u_1^*(t).$$

The last equation means that $Z[t] = \log \tau(t)$, where $\tau(t)$ is a tau-function of the Gelfand-Dikii hierarchy in the A_{q-1} case .

Scale invariance of string equation

Let only the finite number of the parameters $\{t_k^\alpha\}$ be nonzero and enumerated by two integers (m, n). Here $1 \leq m \leq q-1$, $1 \leq n \leq p-1$, p, q are two coprime integers. Then

$$S = \operatorname*{res}_{y=\infty}[Q^{\frac{p+q}{q}} + \sum_{m,n}^{pm-qn>0} \tau_{mn} Q^{\frac{pm-qn}{q}}],$$

It is easy to check that $Q[y, u_\alpha]$ and $S[u_\alpha, \tau_{mn}]$ are quasi-homogeneous functions

$$Q[\rho y, \rho^{r_\alpha} u_\alpha] = \rho^q Q[y, u_\alpha], \qquad S[\rho^{r_\alpha} u_\alpha, \rho^{\sigma_{mn}} \tau_{mn}] = \rho^{p+q} S[u_\alpha, \tau_{mn}].$$

$$r_\alpha = q - \alpha - 1, \qquad \sigma_{mn} = p + q - |pm - qn|.$$

$\delta_{mn} = \frac{\sigma_{mn}}{2q}$ coincide with gravitational dimensions of (p, q) MLG . The function $Z[\tau_{mn}]$ is also a quasi-homogeneous function

$$Z[\rho^{2q\delta_{mn}} \tau_{mn}] = \rho^{2(p+q)} Z[\tau_{mn}].$$

Group of resonance transformations

Since the scaling indices are integer, the following so called resonance relation can take place

$$\sigma_{mn} = \sigma_{k_1 l_1} + \sigma_{k_2 l_2} + \ldots + \sigma_{k_N l_N}.$$

A transformation $\tau_{mn} \to \lambda_{mn}$ of the form

$$\tau_{mn} = \lambda_{mn} + \sum_{k_1,l_1,k_2,l_2} A_{mn}^{k_1 l_1;k_2,l_2} \lambda_{k_1,l_1} \lambda_{k_2,l_2} + \sum_{k_1,l_1,k_2,l_2,k_3,l_3} A_{mn}^{k_1 l_1;k_2,l_2;k_3,l_3} \lambda_{k_1,l_1} \lambda_{k_2}$$

is called the Resonance transformation if the resonance relation is satisfied for each term. It is obvious that

$$\tau_{mn}(\{\rho^{\sigma_{kl}} \lambda_{kl}\}) = \rho^{\sigma_{mn}} \tau_{mn}(\{\lambda_{kl}\}),$$

Resonance transformation respects the homogeneity property of the partition function ,i.e. $Z_L[\{\lambda_{mn}\}] := Z_L[\{\tau_{mn}\}]$

$$Z_L[\{\rho^{\sigma_{mn}} \lambda_{mn}\}] = \rho^{2(p+q)} Z_L[\{\lambda_{mn}\}].$$

After performing the resonance transform

$$
\begin{aligned}
t_{mn} &= \lambda_{mn} + A_{mn}\mu^{\delta_{mn}} + \sum_{m_1,n_1}^{\delta_{m_1n_1}\leq\delta_{mn}} A_{mn}^{m_1n_1}\mu^{\delta_{mn}-\delta_{m_1n_1}}\lambda_{m_1n_1} + \\
&\quad + \sum_{m_1,n_1,m_2,n_2}^{\delta_{m_1n_1}+\delta_{m_2n_2}\leq\delta_{mn}} A_{mn}^{m_1n_1,m_2n_2}\mu^{\delta_{mn}-\delta_{m_1n_1}-\delta_{m_2n_2}}\lambda_{m_1n_1}\lambda_{m_2n_2} + \dots,
\end{aligned}
\tag{1.1}
$$

the action is written in the form

$$
S_L[v_\alpha,\{\lambda_{mn}\}] = S^{(0)}(v_\alpha) + \sum_{m,n}\lambda_{mn}\, S^{(mn)}(v_\alpha) + \sum_{m_1,n_1,m_2,n_2}\lambda_{m_1n_1}\lambda_{m_2n_2}\, S^{(m_1n
$$

where

$$
S^{(0)} = \operatorname*{res}_{y=\infty}\left[Q^{\frac{p+q}{q}} + \sum_{l=1}^{ql\leq p} A_{1l}\ \mu^{\frac{l+1}{2}}\, Q^{\frac{p-ql}{q}}\right],
$$

$$
S^{(mn)} = \operatorname*{res}_{y=\infty}\left[Q^{\frac{pm-qn}{q}} + \sum_{l=2}^{ql\leq pm} A_{ml}^{mn}\ \mu^{\frac{l-n}{2}}\ Q^{\frac{pm-ql}{q}}\right],
$$

The higher coefficients can be also easily written in terms of the coefficients $A_{kl}^{\{m_i n_i\}}$.
The generating function is given by

$$Z_L[\{\lambda_{mn}\}] = \frac{1}{2}\int_0^{\mathbf{v}^*} C_\alpha^{\beta\gamma}(v)\frac{\partial S_L}{\partial v^\beta}\frac{\partial S_L}{\partial v^\gamma}dv^\alpha,$$

where $\mathbf{v}^*$ is defined as a function of the parameters $\{\lambda_{mn}\}$ from the Douglas string equation .

The special solution of String equation

To compute the one-point function which is given by the integral

$$\langle O_{mn} \rangle = \int_0^{v_\alpha^0} C^\alpha_{\beta\gamma} \frac{\partial S^{(0)}}{\partial v^\beta} \frac{\partial S^{(mn)}}{\partial v^\gamma} dv_\alpha,$$

we need to know the upper limit in this integral v_α^0 which is the solution of the string equation for all couplings (except $\lambda_{11} = \mu$) equal to zero

$$v_\alpha^0 = v_\alpha^*(\lambda_{mn}) \Big|_{\lambda_{mn}=0, \lambda_{11}=\mu}.$$

$$\frac{\partial S^{(0)}}{\partial v_\mu} \Big|_{v_\alpha = v_\alpha^0} = 0.$$

We can write $S^{(0)}$ and $S^{(mn)}$in the following explicit form

$$S^{(0)} = -\frac{\theta_{p_0, s+1}}{c_{p_0, s+1}} - \sum_{l=1}^{ql \leq p} A_{1l} \mu^{\frac{l+1}{2}} \frac{\theta_{p_0, s-l}}{c_{p_0, s-l}},$$

$$S^{(mn)} = -\frac{\theta_{p_0 m, sm-n}}{c_{p_0 m, sm-n}} - \sum_{l=n+2}^{ql \leq pm} A^{mn}_{ml} \mu^{\frac{l-n}{2}} \frac{\theta_{p_0 m, sm-l}}{c_{p_0 m, sm-l}}.$$

It was shown by V.B. the following Lemma : if $v_{i>1} = 0$,

$$\begin{cases} k \text{ even :} & \frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,\alpha}\, x_{\lambda,k} \left(-\frac{v_1}{q} \right)^{\frac{k}{2} q}, \\ k \text{ odd :} & \frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,q-\alpha}\, y_{\lambda,k} \left(-\frac{v_1}{q} \right)^{\frac{k-1}{2} q + \lambda}, \end{cases} \tag{1.2}$$

where

$$x_{\alpha,k} = \frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q} + \frac{k}{2})(\frac{k}{2})!} \qquad \text{and} \qquad y_{\lambda,k} = -\frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q} + \frac{k+1}{2})(\frac{k-1}{2})!}.$$

Using this it is not difficult to see that string equation has remarkable solutions of the form $v^0_\alpha = 0$ for $\alpha \neq 1$. the coordinate v^0_1 is a root of the equation

$$\frac{\partial S^{(0)}}{\partial v_{p_0}} = 0, \quad \text{if} \qquad s - \text{odd}.$$

$$\frac{\partial S^{(0)}}{\partial v_{q-p_0}} = 0, \quad \text{if} \qquad s - \text{even}.$$

where after taking derivative we set all v_β for $\beta \neq 1$ to zero.

One point functions

It was shown by V.B., that on the line $v_{\alpha>0} = 0$, for $\alpha \geq \beta \geq \gamma$

$$C_{\alpha\beta\gamma} = (-\frac{v_1}{q})^{\frac{\alpha+\beta+\gamma-q-1}{2}}$$

if $\frac{\alpha+\beta+\gamma-q-1}{2} \in \mathbb{N}_0$ and $(\alpha+\beta-\gamma) \in [1, q-1]$ and is zero otherwise.
If s-odd and $(sm-n)$-even, we get from recursive relations :

$$\langle O_{mn} \rangle = \int_0^{v_1^0} C_{q-1,p_0,p_0 m} \frac{\partial S^{(0)}}{\partial v_{p_0}} \frac{\partial S^{(mn)}}{\partial v_{p_0 m}} dv_1.$$

From the formula for structure constants we get that the correlation function is zero for $m \neq 1$. It follows

$$\langle O_{1n} \rangle = \int_0^{v_1^0} \left(-\frac{v_1}{q}\right)^{p_0-1} \frac{\partial S^{(0)}}{\partial v_{p_0}} \frac{\partial S^{(1n)}}{\partial v_{p_0}} dv_1 = 0.$$

If s odd and $(sm-n)$ odd, from the same formula we get that the correlation function is zero for $p_0(m+1) \neq q$ but then

$$[\langle O_{mn} \rangle] = \frac{p+q}{q} - \delta_{mn} = \frac{sm-n}{2} + \frac{s+1}{2} + \frac{p_0(m+1)}{2q},$$

is integer, the correlator is analytic and should not be considered .

Similarly, for s even and $(sm-n)$ even, we obtain the following consequence of the selection rules

$$\langle O_{1n} \rangle = \int_0^{v_1^0} \Big(-\frac{v_1}{q} \Big)^{q-p_0-1} \frac{\partial S^{(0)}}{\partial v_{q-p_0}} \frac{\partial S^{(1n)}}{\partial v_{q-p_0}} dv_1 = 0.$$

Finally, if s even and $(sm-n)$ odd, we find again that expressions for the one point correlation functions are analitic and hence nonuniversal.

Selection rules for one point functions

Thus to fulfil the selections rules the following one point functions have to be set to zero

$$\langle O_{1n}\rangle = \int_0^{v_1^0} \Big(-\frac{v_1}{q}\Big)^{p_0-1} \frac{\partial S^{(0)}}{\partial v_{p_0}} \frac{\partial S^{(1n)}}{\partial v_{p_0}} dv_1 = 0.$$

is in case s is odd and n is odd.And

$$\langle O_{1n}\rangle = \int_0^{v_1^0} \Big(-\frac{v_1}{q}\Big)^{q-p_0-1} \frac{\partial S^{(0)}}{\partial v_{q-p_0}} \frac{\partial S^{(1n)}}{\partial v_{q-p_0}} dv_1 = 0.$$

if s is even and n is odd.

Simple analysis shows that the number of these equations is equal to the number of the coefficients arising in the Resonance transforms of the first order.
The requirement of absence of the one point functions uniquely fixes them.

Two-point functions

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle = \sum_{\gamma=1}^{q-1} \int_0^{v_1^0} dv_1 \, v_1^{\gamma-1} \, \frac{\partial S^{(m_1 n_1)}}{\partial v_\gamma} \, \frac{\partial S^{(m_2 n_2)}}{\partial v_\gamma}.$$

It follows from the Lemma that $\frac{\partial S^{(mn)}}{\partial v^\gamma} \neq 0$ if one of the following two conditions is satisfied

$$\begin{aligned} &1) \quad \gamma = mp_0 \bmod q \quad \text{and} \quad (sm-n) - \text{even}, \\ &2) \quad \gamma = q - mp_0 \bmod q \quad \text{and} \quad (sm-n) - \text{odd}. \end{aligned}$$

Similary one point case we find four cases where the two point function can be non-zero. In two cases where the first pair satisfies to first condition while the second pair (m_2, n_2) to the second one or vice versa we get the regular expression for the two point function.

Thus we are left with the two options where both pairs satisfy either the first or the second condition .

In the case when both $(sm - n_1)$ and $(sm - n_2)$ are even we get the following requirement for two point functions

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_0^{v_1^0} dv_1 \, v_1^{mp_0 - 1} \, \frac{\partial S^{(mn_1)}}{\partial v_{mp_0}} \, \frac{\partial S^{(mn_2)}}{\partial v_{mp_0}} = 0 \quad \text{if} \quad n_1 \neq n_2 .$$

Making the substitution

$$t = 2\left(\frac{v_1}{v_1^0}\right)^q - 1,$$

and denoting

$$\frac{\partial S^{(mn)}}{\partial v_{mp_0}} = L_{\frac{sm-n}{2}}(t),$$

we find the following consequence of the diagonality condition

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_{-1}^{1} dt \, (1+t)^{\frac{mp_0 - q}{q}} \, L_{\frac{sm-n_1}{2}}(t) L_{\frac{sm-n_2}{2}}(t) = 0 \quad \text{if} \quad n_1 \neq n_2 .$$

So the selection rules for the two-point correlation numbers require that $L_{\frac{sm-n}{2}}$ coincide to Jacobi polynomials

$$\frac{\partial S^{(mn)}}{\partial v_{mp_0}} = \frac{(pm-qn)}{q} P_{\frac{sm-n}{2}}^{(0,\frac{mp_0-q}{q})}(t).$$

In the second case when both $(sm-n_1)$ and $(sm-n_2)$ are odd

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_0^{v_1^0} dv_1\, v_1^{q-mp_0-1}\, \frac{\partial S^{(mn_1)}}{\partial v_{q-mp_0}}\, \frac{\partial S^{(mn_2)}}{\partial v_{q-mp_0}} = 0 \quad \text{if} \quad n_1 \neq n_2.$$

Denoting

$$\frac{\partial S^{(mn)}}{\partial v_{q-mp_0}} = (1+t)^{\frac{mp_0}{q}} L_{\frac{sm-n-1}{2}}(t),$$

we find the following consequence of the diagonality condition

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_{-1}^{1} dt\, (1+t)^{\frac{mp_0}{q}}\, L_{\frac{sm-n_1-1}{2}}(t) L_{\frac{sm-n_2-1}{2}}(t) = 0 \quad \text{if} \quad n_1 \neq n_2.$$

$L_{\frac{sm-n-1}{2}}$ are proportional to Jacobi polynomials

$$\frac{\partial S^{(mn)}}{\partial v_{q-mp_0}} = \frac{(pm-qn)}{q} (1+t)^{\frac{mp_0}{q}} P_{\frac{sm-n-1}{2}}^{(0,\frac{mp_0}{q})}(t)$$

At last inserting these explicit expressions for derivaties of $S^{(mn)}$ to the vanishing equations for one point functions we arrive to equations

$$\langle O_{1n}\rangle = \int_{-1}^{1}(1+t)^{\frac{p_0-q}{q}}\frac{\partial S^{(0)}}{\partial v_{p_0}}(t)P_{\frac{s-n}{2}}^{(0,\frac{p_0-q}{q})}(t)dt = 0.$$

in case s is odd and n is odd and greater than 1. And

$$\langle O_{1n}\rangle = \int_{-1}^{1}(1+t)^{\frac{p_0}{q}}\frac{\partial S^{(0)}}{\partial v_{q-p_0}}(t)P_{\frac{s-n-1}{2}}^{(0,\frac{p_0}{q})}(t)dt = 0.$$

in case s is even and n is odd and greater than 1.
Taking to the account the these equations, the order the polynomials $\frac{\partial S^{(0)}}{\partial v_{p_0}}$ and $\frac{\partial S^{(0)}}{\partial v_{p_0}}$ and the string equations , we obtain

$$\frac{\partial S^{(0)}}{\partial v_{p_0}} = \frac{(p+q)}{q}[P_{\frac{s+1}{2}}^{(0,\frac{p_0-q}{q})}(t) - P_{\frac{s-1}{2}}^{(0,\frac{p_0-q}{q})}(t)]$$

if s is odd and

$$\frac{\partial S^{(0)}}{\partial v_{q-p_0}} = \frac{(p+q)}{q}(1+t)^{\frac{p_0}{q}}[P_{\frac{s}{2}}^{(0,\frac{p_0}{q})}(t) - P_{\frac{s-2}{2}}^{(0,\frac{p_0}{q})}(t)]$$

if s is even

Conclusions

- I have described relation between the approach to (p, q) models of Minimal Liouville gravity based on the Douglas string equation, on one hand, and the Frobenius manifolds of A_{q-1} type on the other.
- Using this relation and some special properties of the flat coordinates on the Frobenius manifold, we have found the appropriate solution of the Douglas string equation.
- It was shown that the appropriate solution is consistent with the basic requirements of the conformal selection rules arising on the levels of one- and two-point correlation functions.
- It would be interesting to understand the Frobenius manifold of what type is inderlie $W(N)$ gravity for case $N > 2$.

Thank you for your attention!