Convergence of Measures in a Non-Archimedean Stochastic Setting

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International Conference on $p$-ADIC MATHEMATICAL PHYSICS AND ITS APPLICATIONS, Belgrade
We will define a random walk and show convergence to a Brownian motion given by Varadarajan [VSV97]. The convergence is in the sense of weak convergence of probability measures.

Can be used to prove that certain quantum Hamiltonians can be approximated by finite quantum systems.
Schrödinger operator

- \( H = P^\alpha + V \), acting on \( L^2(X^d) \) where \( X = \mathbb{R} \) or \( X = K \), a local field. For simplicity we will work in dimension 1 (\( d = 1 \)).

- \( P = \mathcal{F}^{-1} Q \mathcal{F} \), \( Q = \) multiplication by the absolute value of the coordinate: \( (Qf)(x) = |x| f(x) \).

- \( \alpha \) : a positive real number. If \( X = \mathbb{R} \) and \( \alpha = 2 \), we recover the Laplacian: \( P^2 = -\Delta \).

- \( V \) (potential): multiplication by a continuous function \( v \) which goes to infinity at infinity, implying discrete spectrum for \( H \).
**Local fields**

- **$K$:** a local field, i.e., a non-discrete, totally disconnected, locally compact field.

Two main types of local fields:
- **char $K = 0$:** $K$ is a finite extension of $\mathbb{Q}_p$ for some $p$.
- **char $K > 0$:** $K$ is isomorphic to the field of Laurent series over a finite field $\mathbb{F}_q$, where $q = p^f$, $p =$ char $K$.

- **$| \cdot |$:** canonical absolute value, induced by the Haar measure. It defines the topology, and is non-Archimedean.

- **$O = \{ x \in K : |x| \leq 1 \}$:** a compact sub-ring of $K$ called the ring of integers. It is a discrete valuation ring, i.e., a principal ideal domain with a unique maximal ideal.

- **$P = \{ x \in K : |x| < 1 \}$:** the unique maximal ideal of $O$, called the prime ideal. We have $P = \beta O$ for some $\beta \in O$. Such a $\beta$ is called a **uniformizer**.

- **$O/P$ is a finite field.** If $q = p^f$ is the number of elements in $O/P$, then $|\beta| = 1/q$ for any uniformizer $\beta$. 
If $S$ is a complete set of representatives for $O/P$, every $x \in K$ can be written uniquely in the form

$$x = \beta^{-m}(x_0 + x_1 \beta + x_2 \beta^2 + \cdots),$$

where $m \in \mathbb{Z}$, $x_j \in S$, $x_0 \not\in P$. With $x$ written in this form, we have $|x| = q^m$. So \(\text{range}(\cdot |) = \{q^N : N \in \mathbb{Z}\}.\)
Fix a Haar measure $\mu$ such that $\mu(O) = 1$, and define the Fourier transform $\mathcal{F}$ by

$$\mathcal{F}f(\xi) = \int_K f(x) \chi(-x\xi) \, dx$$

for a suitably chosen additive character $\chi$. For our setup it will be essential to work with a character of rank 0.\(^1\)

\(^{1}\)rank$(\chi) = \max\{r \in \mathbb{Z} : \chi|_{B_r} \equiv 1\}$, $B_r = \{x \in K : |x| \leq q^r\}$. 
Finite model

- $B_n = \beta^{-n}O =$ ball of radius $q^n$: an open additive subgroup of $K$.
- $G_n = B_n/B_{-n}$: a finite group with $q^{2n}$ elements ($n \geq 0$).
- Each element of $G_n$ has a unique representative of the form
  
  $a_{-n}\beta^{-n} + a_{-n+1}\beta^{-n+1} + \cdots + a_{-1}\beta^{-1} + a_0 + a_1\beta + \cdots$
  
  $+ a_{n-2}\beta^{n-2} + a_{n-1}\beta^{n-1}$.

  We denote the set of these representatives by $X_n$, and give it the group structure inherited from $G_n$.

- Haar measure $\mu_n$ on $G_n$:
  
  $\mu_n(\{x + H_n\}) = \mu(x + H_n) = \mu(H_n) = q^{-n}$, $\{x + H_n\} \in G_n$.

  So each point $\{x + H_n\}$ of $G_n$ has mass $q^{-n}$, and the total mass of $G_n$ is $q^{2n} \cdot q^{-n} = q^n$.

$^2$For convenience we often write $H_n = B_{-n}$; so for instance we have $G_n = H_{-n}/H_n$. 
\textbullet \ L^2\text{-isometric imbedding } L^2(G_n) \to L^2(K):

\[ \mathbf{1}_{\{x+H_n\}} \in L^2(G_n) \mapsto \mathbf{1}_{x+H_n} \in L^2(K). \]

An operator on \( L^2(G_n) \) is regarded as an operator on \( L^2(K) \) via this imbedding, by setting it equal to 0 on the orthogonal complement of the image of \( L^2(G_n) \).
Important subspaces of $L^2(K)$

- $C_n = \{f \in L^2(K) | \text{supp}(f) \subset B_n\}$. The corresponding orthogonal projection is denoted by $C_n$ and is given by: $C_n f = 1_{B_n} f$.

- $S_n = \{f \in L^2(K) | f \text{ is locally constant of index } \leq q^{-n}\}$. The corresponding orthogonal projection is denoted by $S_n$ and is given by:

$$ (S_n f)(x) = q^n \int_{H_n} f(x + y) \, dy = \frac{1}{\mu(H_n)} \int_{H_n} f(x + y) \, dy = \text{ave}(f, n, x), $$

where we have introduced the notation $\text{ave}(f, n, x)$ for the average value of $f$ over $x + H_n$.

- $D_n = C_n \cap S_n$. The corresponding orthogonal projection is denoted by $D_n$.

$L^2(G_n)$ is mapped onto $D_n$ via the isometric imbedding mentioned above. Thus $L^2(G_n)$ can be thought of as the set of functions on $K$ which have support in $B_n$ and which are invariant under translation by elements of $H_n (= B_{-n})$. 
Commutation relations

\[ D_n = C_n S_n = S_n C_n \]
\[ \mathcal{F} C_n = S_n, \quad \mathcal{F} S_n = C_n, \quad \mathcal{F} D_n = D_n \]
\[ \mathcal{F} C_n = S_n \mathcal{F}, \quad \mathcal{F} S_n = C_n \mathcal{F}, \quad \mathcal{F} D_n = D_n \mathcal{F} \]
Fourier transform at the finite level

Let as before $\chi$ be a rank zero character on $K$. The bi-character $(x, y) \mapsto \chi(xy)$ descends to a non-degenerate bi-character on $G_n = B_n / B_{-n}$, thus the natural choice for an $L^2$-isometric Fourier transform on $X_n \cong G_n$ is

$$(F_n f)(x) = \frac{1}{\sqrt{|X_n|}} \sum_{y \in X_n} f(y) \chi(-xy) = q^{-n} \sum_{y \in X_n} f(y) \chi(-xy), \quad x \in X_n, \quad f \in L^2(X_n).$$

**Crucial fact:**
The Fourier transform $\mathcal{F}$ on $K$ descends to the Fourier transform $\mathcal{F}_n$ on $X_n$:

$$\mathcal{F}|_{D_n} = \mathcal{F}_n, \text{ i.e., } \mathcal{F}_n = \mathcal{F} D_n = D_n \mathcal{F}.$$
Dynamical operators at the finite level

The finite operators are obtained through compression by the projection $D_n$:

$$V_n = D_n V D_n, \quad Q_n = D_n Q D_n, \quad P_n = D_n P D_n = F_n^{-1} Q_n F_n$$

We have, for $f \in L^2(G_n)$:

$$(V_n f)(x) = v_n(x) f(x), \quad v_n(x) = \frac{1}{\mu(H_n)} \int_{x+H_n} v(h) \, dh$$

$$(Q_n f)(x) = r_n(x) f(x), \quad r_n(x) = \frac{1}{\mu(H_n)} \int_{x+H_n} |h| \, dh$$

$$= \begin{cases} 
|x|, & |x| > q^{-n} \\
\text{ave}(|x|, n, 0), & |x| \leq q^{-n}
\end{cases}$$

$$H_n = P_n^\alpha + V_n = F_n^{-1} Q_n^\alpha F_n + V_n \text{ (finite Hamiltonian.)}$$
Let $D[0, T]$ be the set of Skorokhod functions on $[0, T]$, that is, the space of all $K$ valued functions $\omega$ on $[0, T]$ which are right continuous and where the left limit exists:

- $\omega(t^+) = \omega(t)$ for $0 \leq t < T$.
- $\omega(t^-)$ exists for $0 < t \leq T$.
- $\omega(T^-) = \omega(T)$.
Let $\omega_1, \omega_2 \in D[0, T]$. Then a metric is defined by

$$d(\omega_1, \omega_2) = \inf_{\lambda \in \Lambda} \{ ||\lambda - I||_{\infty}, ||\omega_1 - \omega_2 \circ \lambda||_{\infty} \},$$

where $\Lambda$ is the set of all strictly increasing, continuous mappings of $[0, T]$ into itself and $I$ is the identity function. The Skorokhod space is separable, and it is complete in an equivalent metric.
The probability density for Brownian motion $\sigma_t$ is given by Varadarajan [VSV97]:

$$\rho_t(x) = e^{-t|x|^\alpha}, \quad \sigma_t(x) = \left[ \mathcal{F}^{-1} \rho_t \right](x).$$

We define it similarly for our finite models:

$$\rho_{n,t}(x) = c_{n,t} e^{-tr_{n}(x)\alpha}, \quad \sigma_{n,t}(x) = \left[ \mathcal{F}^{-1}_{n} \rho_{n,t} \right](x),$$

where

$$r_n(x) = \begin{cases} |x|, & |x| > q^{-n} \\ \text{ave}(|x|, n, 0), & |x| \leq q^{-n}, \end{cases}$$

and $c_{n,t}$ is a positive number, adjusted so that $\rho_{n,t}(0) = 1$. The densities $\sigma_t$ and $\sigma_{n,t}$ are positive with integral equal to 1.
Fix $k$ time points $t_1, \ldots, t_k$. Define the measure $P_{a_n}^n$ on cylinder sets by

$$P_{a_n}^n(\omega(t_i) \in J_i) = \sum_{b_i \in J_i \cap X_n} \sigma_{n,t_1}(b_1-a_n) \cdots \sigma_{n,t_k-t_{k-1}}(b_k-b_{k-1}) q^{-nk},$$

where $J_i$ are Borel sets. It satisfies the consistency conditions:

$$P_{a_n}^{n,t_\kappa(1), \ldots, t_\kappa(k)}(J_1 \times \cdots \times J_k) = P_{a_n}^{n,t_1, \ldots, t_k}(J_{\kappa^{-1}(1)} \times \cdots \times J_{\kappa^{-1}(k)})$$

for any permutation $\kappa$ on $\{1, \ldots, k\}$, and where $J_i \ (1 \leq i \leq k)$ are Borel sets in $K$.

$$P_{a_n}^{n,t_1, \ldots, t_1}(J_1 \times \cdots \times J_k) = P_{a_n}^{n,t_1, \ldots, t_k, t_{k+1}, \ldots, t_{k+m}}(J_1 \times \cdots \times J_k \times K \times \cdots \times K),$$

$m$ times

where $J_i \ (1 \leq i \leq k)$ are Borel sets in $K$.

By Kolmogorov, we get a probability measure $P_{a_n}^n$ on the set of all paths.
Let $X_t$ be the stochastic process given by $X_t(\omega) = \omega(t)$.

**Theorem**

Let $\mathbf{P}_0$ be a measure on the space of all paths on $[0, T]$. If there exist constants $C, a, b, c > 0$ such that for all $0 \leq t_1 < t_2 < t_3 \leq T$,

$$E_{\mathbf{P}_0}(|X_{t_2} - X_{t_1}|^a|X_{t_3} - X_{t_2}|^b) \leq C(t_3 - t_1)^{1+c}.$$

Then there exists a unique probability measure $\mathbf{P}$ on $D[0, T]$ which has the same finite-dimensional distributions as $\mathbf{P}_0$.

We have that

$$E_{\mathbf{P}_{a_n}}(|X_{t_2} - X_{t_1}|^k|X_{t_3} - X_{t_2}|^k) \leq A(t_3 - t_1)^{2k/\alpha},$$

so the Chentsov criterion is satisfied for $\alpha/2 < k < \alpha$. Thus $\mathbf{P}_{a_n}$ gives full measure to all paths in $D[0, T]$ starting at $a_n$. Furthermore, it has support on the paths on the grid $X_n$. 
The measure $P_a$ from [VSV97] is constructed in a similar way: Fix $k$ time points $t_1, \ldots, t_k$. Define the measure $P_a$ on cylinder sets by

$$P_a(\omega(t_i) \in J_i) = \int_{J_1} \cdots \int_{J_k} \sigma_{t_1}(b_1-a) \cdots \sigma_{t_{k-1}t_{k-1}}(b_k-b_{k-1}) \, db_k \cdots db_1,$$

where $J_i$ ($1 \leq i \leq k$) are Borel sets.

By using Kolmogorov and Chentsov we get the measure $P_a$ on $D[0, T]$. 
Definition

Let \((S, \mathcal{S})\) be a measure space. A sequence of probability measures \(P_k\) is said to converge weakly to \(P\) if
\[
\int f \, dP_k \to \int f \, dP
\]
for all bounded continuous functions \(f\) on \(S\). This will be denoted by \(P_k \Rightarrow P\).

The Portmanteau Theorem

The following are equivalent:

1. \(P_k \Rightarrow P\).
2. \[
\int f \, dP_k \to \int f \, dP
\]
   for all bounded, uniformly continuous functions \(f\).
3. \(P_k(A) \to P(A)\) for all measurable sets \(A\) with \(P(\partial A) = 0\).

We wish to prove that \(P_{a_n} \Rightarrow P_a\) when \(a_n \to a\).
Weak Convergence in $D[0, T]$

**Definition**

A sequence of probability measures $P_m$ is tight if there for any $\epsilon > 0$ exists a compact set $J$ such that $P_m(J) > 1 - \epsilon$ for all $m$.

**Theorem**

*Suppose that $P_m, P$ are probability measures on $D[0, T]$ such that*

- $P_m^{t_1, \ldots, t_N} \Rightarrow P^{t_1, \ldots, t_N}$ for all $t_1, \ldots, t_N$ in $[0, T]$.
- *The measures $P_m$ are tight: There are constants $C, a, b, c > 0$ such that for all $m$ and $0 \leq t_1 < t_2 < t_3 \leq T$,

$$E_{P_m}(|X_{t_2} - X_{t_1}|^a |X_{t_3} - X_{t_2}|^b) \leq C(t_3 - t_1)^{1+c},$$

*then $P_m \Rightarrow P$.*

Since our estimate for Chentsov’s criterion is independent of $n$, we know that the measures $P_{an}$ are tight.
From Billingsley:

**Theorem**

Let $\mathbf{P}$ be a probability measure, and let $\mathbf{P}_m$ be a sequence of probability measures. Suppose that
- $\mathcal{A}$ is a $\pi$-system
- Every open set is a countable union of elements in $\mathcal{A}$.

If $\mathbf{P}_m(A) \to \mathbf{P}(A)$ for all $A \in \mathcal{A}$, then $\mathbf{P}_m \Rightarrow \mathbf{P}$.

To prove that $\mathbf{P}_{a_n}^{n,t_1,...,t_k} \Rightarrow \mathbf{P}_a^{t_1,...,t_k}$, we only have to prove that

$$
\mathbf{P}_{a_n}^{n,t_1,...,t_k}(\omega(t_i) \in B_i) \Rightarrow \mathbf{P}_a^{t_1,...,t_k}(\omega(t_i) \in B_i)
$$

for all balls $B_i$, $1 \leq i \leq k$.

**Theorem**

We have that

$$
\mathbf{P}_{a_n}^n \Rightarrow \mathbf{P}_a
$$

when $a_n \to a$. 

The conditional measure $P_{a,b,T}$ is defined in [VSV97] by

$$P_{a,b,T}(B) = P_a(B|\omega(T) = b).$$

Since we are conditioning on a set of measure 0, it is defined by probability densities. The Chentsov criterion holds [VSV97], so it gives full measure to the Skorokhod paths which start at $a$ and end up at $b$ at time $T$, and it models a Brownian motion in $K$ going from $a$ to $b$ in time $T$. 
We define the conditional measure $P^n_{a_n,b_n,T}$ of a Borel set $B$ by

$$P^n_{a_n,b_n,T}(B) = \frac{P^n_{a_n}(B \cap (\omega(T) = b_n))}{P^n_{a_n}(\omega(T) = b_n)}.$$ 

It gives full measure to the Skorokhod paths which start at $a_n$ and end up at $b_n$ at time $T$.

We wish to prove that $P^n_{a_n,b_n,T} \Rightarrow P_{a,b,T}$ when $a_n \to a$ and $b_n \to b$. The proof for convergence of the finite-dimensional distributions goes exactly as for the unconditioned measures. The difficult part is tightness, and the proof is similar to the one in [DVV94].
Modulus of Continuity

Definition

The "modulus of continuity" for a Skorokhod path $\omega$ is

$$m(\omega : \delta) = \sup_{s_1 < s < s_2} \min \{|\omega(s_2) - \omega(s)|, |\omega(s) - \omega(s_1)|\}.$$

Theorem

Let $P_k$ be a sequence of probability measures on $D[0, T]$.

TFAE:

- The sequence of measures $P_k$ is tight.
- For every $\eta > 0$,

$$\lim_{\delta \to 0} P_k(\{\omega : m(\omega : \delta) \geq \eta\}) = 0$$

uniformly in $k$. 
Comparison with Unconditioned Measure

Define

\[ m_1(\omega : \delta) = \sup_{s_1 < s < s_2, s_2 - s_1 < \delta, s_2 \leq 3T/4} \min \{|\omega(s_2) - \omega(s)|, |\omega(s) - \omega(s_1)|\} \]

With \( A_1 = \{\omega : m_1(\omega : \delta) \geq \eta\} \), we first prove that

\[ \lim_{\delta \to 0} P_{n, a_n, b_n, T}(A_1) = 0 \]

uniformly in \( n \). The advantage here is that \( 3T/4 \) is far away from \( T \) where we are conditioning.

\[ P_{n, a_n, b_n, T}(A_1) \leq CP_{a_n}(A_1) \]

for some constant \( C \). The measures \( P_{a_n} \) are tight so for every \( \eta > 0 \),

\[ \lim_{\delta \to 0} P_{n, a_n, b_n, T}(A_1) \leq C \lim_{\delta \to 0} P_{a_n}(A_1) = 0 \]

uniformly in \( n \).
We can do the same over the interval \([T/4, T]\) by time-reversal:

\[ x^*(s) = x(T - s - 0), \quad 0 \leq s < T \]

and \(x^*(T) = x(0)\). The time reversal is a Borel function which is involutive. Define the probability measure \(P^*(E) = P(E^*)\). With this definition

\[ (P^n_{a_n,b_n,T})^* = P^n_{b_n,a_n,T} \]

This comes from stochastic continuity, which means that if \(s_i < s\), then

\[ P^n_{a_n,b_n,T}(\omega : |X_s - X_{s_i}| > \epsilon) \to 0. \]

as \(s_i \to s\).

**Theorem**

We get that \(P^n_{a_n,b_n,T} \Rightarrow P_{a,b,T}\) when \(a_n \to a\) and \(b_n \to b\).
We have proved that \( P_{a_n, b_n, T} \Rightarrow P_{a, b, T} \) when \( a_n \to a \) and \( b_n \to b \). The convergence is also uniform when \( a \) and \( b \) vary in a compact set: If \( g \) is any bounded continuous function on \( D[0, T] \), then

\[
\int_{D[0, T]} g(\omega) \, dP^n_{a_n, b_n, T}(\omega) \to \int_{D[0, T]} g(\omega) \, dP_{a, b, T}(\omega)
\]

is uniform with respect to \( a \) and \( b \) varying in a compact set.
\[
e^{-tP_n^\alpha} g(x) = e^{-t\mathcal{F}_n^{-1}Q_n^\alpha} \mathcal{F}_n g(x) \\
= \mathcal{F}_n^{-1} e^{-tQ_n^\alpha} \mathcal{F}_n g(x) = [\mathcal{F}_n^{-1} e^{-t\mathcal{F}_n^\alpha} \ast g](x) = [\sigma_n, t \ast g](x), \\
e^{-(t/N)Q_n} e^{-(t/N)V_n} g(x) = \int_{G_n} \sigma_{n, t/N}(y-x) e^{-(t/N)v_n(y)} g(y) \, d\mu_n(y) \\
\left( e^{-(t/N)Q_n} e^{-(t/N)V_n} \right)^N g(x) \\
= \int_{G_n^N} \sigma_{n, t/N}(x-x_1) \cdots \sigma_{n, t/N}(x_{N-1} - x_N) \\
\cdot e^{-(t/N)\sum_{i=1}^N v_n(x_i)} g(x_N) \, d\mu_n(x_1) \cdots d\mu_n(x_N)
\]

By using the Trotter product formula one obtains Feynman-Kac.

\[
e^{-tH_n}(j_n, k_n) = \int_{D[0,t]} e^{-\int_0^t v_n(\omega(s)) \, ds} \, dP_{j_n,k_n,t}(\omega) \cdot \sigma_n, t(k_n - j_n) q^{-n}.
\]