Convergence of Measures in a Non-Archimedean Stochastic Setting

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Setting

- We will define a random walk and show convergence to a Brownian motion given by Varadarajan [VSV97]. The convergence is in the sense of weak convergence of probability measures.
- Can be used to prove that certain quantum Hamiltonians can be approximated by finite quantum systems.

Setting

Schrödinger operator

- $H = P^\alpha + V$, acting on $L^2(X^d)$ where $X = \mathbf{R}$ or $X = K$, a local field. For simplicity we will work in dimension 1 ($d = 1$).
- $P = \mathcal{F}^{-1} Q \mathcal{F}$, $Q =$ multiplication by the absolute value of the coordinate: $(Qf)(x) = |x| f(x)$.
- α : a positive real number. If $X = \mathbf{R}$ and $\alpha = 2$, we recover the Laplacian: $P^2 = -\Delta$.
- V (potential): multiplication by a continuous function v which goes to infinity at infinity, implying discrete spectrum for H.

Local fields

- K: a local field, i.e., a non-discrete, totally disconnected, locally compact field.
- Two main types of local fields:
  - char $K = 0$: K is a finite extension of $\mathbf{Q}_p$ for some p.
  - char $K > 0$: K is isomorphic to the field of Laurent series over a finite field $\mathbf{F}_q$, where $q = p^f$, $p =$ char K.
- $|\cdot|$: canonical absolute value, induced by the Haar measure. It defines the topology, and is non-Archimedean.
- $O = \{x \in K : |x| \le 1\}$: a compact sub-ring of K called the ring of integers. It is a discrete valuation ring, i.e., a principal ideal domain with a unique maximal ideal.
- $P = \{x \in K : |x| < 1\}$: the unique maximal ideal of O, called the prime ideal. We have $P = \beta O$ for some $\beta \in O$. Such a β is called a *uniformizer*.
- O/P is a finite field. If $q = p^f$ is the number of elements in O/P, then $|\beta| = 1/q$ for any uniformizer β.

Local fields (cont'd)

- If S is a complete set of representatives for O/P, every $x \in K$ can be written uniquely in the form

$$x = \beta^{-m}(x_0 + x_1\beta + x_2\beta^2 + \cdots),$$

where $m \in \mathbf{Z}$, $x_j \in S$, $x_0 \notin P$. With x written in this form, we have $|x| = q^m$. So range$(|\cdot|) = \{q^N \colon N \in \mathbf{Z}\}$.

Haar measure, characters and Fourier transform

Fix a Haar measure μ such that $\mu(O) = 1$, and define the Fourier transform $\mathcal{F}$ by

$$(\mathcal{F}f)(\xi) = \int_K f(x)\chi(-x\xi)\,dx$$

for a suitably chosen additive character χ. For our setup it will be essential to work with a character of rank 0.[1]

[1] $\text{rank}(\chi) = \max\{r \in \mathbf{Z} : \chi|_{B_r} \equiv 1\}$, $B_r = \{x \in K : |x| \leq q^r\}$.

Finite model

- $B_n = \beta^{-n}O$ = ball of radius q^n: an open additive subgroup of K.
- $G_n = B_n/B_{-n}$: a finite group with q^{2n} elements ($n \geq 0$).[2]
- Each element of G_n has a unique representative of the form

$$a_{-n}\beta^{-n} + a_{-n+1}\beta^{-n+1} + \cdots + a_{-1}\beta^{-1} + a_0 + a_1\beta + \cdots + a_{n-2}\beta^{n-2} + a_{n-1}\beta^{n-1}.$$

  We denote the set of these representatives by X_n, and give it the group structure inherited from G_n.
- Haar measure μ_n on G_n:
  $\mu_n(\{x + H_n\}) = \mu(x + H_n) = \mu(H_n) = q^{-n}$, $\{x + H_n\} \in G_n$.
  So each point $\{x + H_n\}$ of G_n has mass q^{-n}, and the total mass of G_n is $q^{2n} \cdot q^{-n} = q^n$.

[2] For convenience we often write $H_n = B_{-n}$; so for instance we have $G_n = H_{-n}/H_n$.

Finite model (cont'd)

- L^2-isometric imbedding $L^2(G_n) \to L^2(K)$:

$$\mathbf{1}_{\{x+H_n\}} \in L^2(G_n) \mapsto \mathbf{1}_{x+H_n} \in L^2(K).$$

An operator on $L^2(G_n)$ is regarded as an operator on $L^2(K)$ via this imbedding, by setting it equal to 0 on the orthogonal complement of the image of $L^2(G_n)$.

Important subspaces of $L^2(K)$

- $\mathcal{C}_n = \{f \in L^2(K) | \operatorname{supp}(f) \subset B_n\}$. The corresponding orthogonal projection is denoted by C_n and is given by: $C_n f = \mathbf{1}_{B_n} f$.
- $\mathcal{S}_n = \{f \in L^2(K) | f$ is locally constant of index $\leq q^{-n}\}$. The corresponding orthogonal projection is denoted by S_n and is given by: $(S_n f)(x) = q^n \int_{H_n} f(x+y)\,dy = \frac{1}{\mu(H_n)} \int_{H_n} f(x+y)\,dy = \operatorname{ave}(f,n,x)$, where we have introduced the notation $\operatorname{ave}(f,n,x)$ for the average value of f over $x + H_n$.
- $\mathcal{D}_n = \mathcal{C}_n \cap \mathcal{S}_n$. The corresponding orthogonal projection is denoted by D_n.

$L^2(G_n)$ is mapped onto $\mathcal{D}_n$ via the isometric imbedding mentioned above. Thus $L^2(G_n)$ can be thought of as the set of functions on K which have support in B_n and which are invariant under translation by elements of $H_n \,(= B_{-n})$.

Commutation relations

$$D_n = C_n S_n = S_n C_n$$

$$\mathcal{F}\mathcal{C}_n = \mathcal{S}_n, \quad \mathcal{F}\mathcal{S}_n = \mathcal{C}_n, \quad \mathcal{F}\mathcal{D}_n = \mathcal{D}_n$$

$$\mathcal{F}C_n = S_n\mathcal{F}, \quad \mathcal{F}S_n = C_n\mathcal{F}, \quad \mathcal{F}D_n = D_n\mathcal{F}$$

Fourier transform at the finite level

Let as before χ be a rank zero character on K. The bi-character $(x, y) \mapsto \chi(xy)$ descends to a non-degenerate bi-character on $G_n = B_n/B_{-n}$, thus the natural choice for an L^2-isometric Fourier transform on $X_n \cong G_n$ is

$$(\mathcal{F}_n f)(x) = \frac{1}{\sqrt{|X_n|}} \sum_{y \in X_n} f(y)\chi(-xy)$$
$$= q^{-n} \sum_{y \in X_n} f(y)\chi(-xy), \quad x \in X_n, \quad f \in L^2(X_n).$$

Crucial fact:
The Fourier transform $\mathcal{F}$ on K descends to the Fourier transform $\mathcal{F}_n$ on X_n:

$$\mathcal{F}|_{\mathcal{D}_n} = \mathcal{F}_n, \text{ i.e.}, \mathcal{F}_n = \mathcal{F} D_n = D_n \mathcal{F}.$$

Dynamical operators at the finite level

The finite operators are obtained through compression by the projection D_n:

$$V_n = D_n V D_n, \quad Q_n = D_n Q D_n, \quad P_n = D_n P D_n = \mathcal{F}_n^{-1} Q_n \mathcal{F}_n$$

We have, for $f \in L^2(G_n)$:

$$(V_n f)(x) = v_n(x) f(x), \quad v_n(x) = \frac{1}{\mu(H_n)} [\int_{x+H_n} v(h)\, dh]$$

$$(Q_n f)(x) = r_n(x) f(x), \quad r_n(x) = \frac{1}{\mu(H_n)} [\int_{x+H_n} |h|\, dh]$$

$$= \begin{cases} |x|, & |x| > q^{-n} \\ \text{ave}(|x|, n, 0), & |x| \le q^{-n} \end{cases}$$

$$H_n = P_n^\alpha + V_n = \mathcal{F}_n^{-1} Q_n^\alpha \mathcal{F}_n + V_n \text{ (finite Hamiltonian.)}$$

Skorokhod Space over K

Let $D[0, T]$ be the set of Skorokhod functions on $[0, T]$, that is, the space of all K valued functions ω on $[0, T]$ which are right continuous and where the left limit exists:

- $\omega(t^+) = \omega(t)$ for $0 \leq t < T$.
- $\omega(t^-)$ exists for $0 < t \leq T$.
- $\omega(T^-) = \omega(T)$.

Skorokhod Space Topology

Let $\omega_1, \omega_2 \in D[0, T]$. Then a metric is defined by

$$d(\omega_1, \omega_2) = \inf_{\lambda \in \Lambda} \{ ||\lambda - I||_\infty, ||\omega_1 - \omega_2 \circ \lambda||_\infty \},$$

where Λ is the set of all strictly increasing, continuous mappings of $[0, T]$ into itself and I is the identity function. The Skorokhod space is separable, and it is complete in an equivalent metric.

Probability Density

The probability density for Brownian motion σ_t is given by Varadarajan [VSV97]:

$$\rho_t(x) = e^{-t|x|^\alpha}, \quad \sigma_t(x) = \left[\mathcal{F}^{-1}\rho_t\right](x).$$

We define it similarly for our finite models:

$$\rho_{n,t}(x) = c_{n,t}e^{-tr_n(x)^\alpha}, \quad \sigma_{n,t}(x) = \left[\mathcal{F}_n^{-1}\rho_{n,t}\right](x),$$

where

$$r_n(x) = \begin{cases} |x|, & |x| > q^{-n} \\ \text{ave}(|x|, n, 0), & |x| \leq q^{-n} \end{cases},$$

and $c_{n,t}$ is a positive number, adjusted so that $\rho_{n,t}(0) = 1$. The densities σ_t and $\sigma_{n,t}$ are positive with integral equal to 1.

Kolmogorov

Fix k time points $t_1, ..., t_k$. Define the measure $P^n_{a_n}$ on cylinder sets by

$$\mathbf{P}^n_{a_n}(\omega(t_i) \in J_i) = \sum_{\substack{b_i \in J_i \cap X_n \\ 1 \le i \le k}} \sigma_{n,t_1}(b_1 - a_n) \cdots \sigma_{n,t_k - t_{k-1}}(b_k - b_{k-1}) q^{-nk},$$

where J_i are Borel sets. It satisfies the consistency conditions:

$$\mathbf{P}^{n,t_{\kappa(1)},\ldots,t_{\kappa(k)}}_{a_n}(J_1 \times \cdots \times J_k) = \mathbf{P}^{n,t_1,\ldots,t_k}_{a_n}(J_{\kappa^{-1}(1)} \times \cdots \times J_{\kappa^{-1}(k)})$$

for any permutation κ on $\{1, ..., k\}$, and where J_i $(1 \le i \le k)$ are Borel sets in K.

$$\mathbf{P}^{n,t_1,\ldots,t_1}_{a_n}(J_1 \times \cdots \times J_k) = \mathbf{P}^{n,t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}_{a_n}(J_1 \times \cdots \times J_k \times \underbrace{K \times \cdots \times K}_{m \text{ times}}),$$

where J_i $(1 \le i \le k)$ are Borel sets in K.

By Kolmogorov, we get a probability measure $P^n_{a_n}$ on the set of all paths.

Chentsov

Let X_t be the stochastic process given by $X_t(\omega) = \omega(t)$.

Theorem

Let $\mathbf{P}_0$ be a measure on the space of all paths on $[0, T]$. If there exist constants $C, a, b, c > 0$ such that for all $0 \leq t_1 < t_2 < t_3 \leq T$,

$$E_{\mathbf{P}_0}(|X_{t_2} - X_{t_1}|^a |X_{t_3} - X_{t_2}|^b) \leq C(t_3 - t_1)^{1+c}.$$

Then there exists a unique probability measure $\mathbf{P}$ on $D[0, T]$ which has the same finite-dimensional distributions as $\mathbf{P}_0$.

We have that

$$E_{\mathbf{P}^n_{a_n}}(|X_{t_2} - X_{t_1}|^k |X_{t_3} - X_{t_2}|^k) \leq A(t_3 - t_1)^{2k/\alpha},$$

so the Chentsov criterion is satisfied for $\alpha/2 < k < \alpha$. Thus $\mathbf{P}^n_{a_n}$ gives full measure to all paths in $D[0, T]$ starting at a_n. Furthermore, it has support on the paths on the grid X_n.

The Measure P_a

The measure $\mathbf{P}_a$ from [VSV97] is constructed in a similar way: Fix k time points $t_1, ..., t_k$. Define the measure $\mathbf{P}_a$ on cylinder sets by

$$\mathbf{P}_a(\omega(t_i) \in J_i) = \int_{J_1} \cdots \int_{J_k} \sigma_{t_1}(b_1 - a) \cdots \sigma_{t_k - t_{k-1}}(b_k - b_{k-1})\, db_k \cdots db_1,$$

where J_i $(1 \leq i \leq k)$ are Borel sets.
By using Kolmogorov and Chentsov we get the measure $\mathbf{P}_a$ on $D[0, T]$.

Weak Convergence of Measures

Definition

Let $(S, \mathcal{S})$ be a measure space. A sequence of probability measures $\mathbf{P}_k$ is said to converge weakly to $\mathbf{P}$ if $\int f \, d\mathbf{P}_k \to \int f \, d\mathbf{P}$ for all bounded continuous functions f on S. This will be denoted by $\mathbf{P}_k \Rightarrow \mathbf{P}$.

The Portmanteau Theorem

The following are equivalent:

- $\mathbf{P}_k \Rightarrow \mathbf{P}$.
- $\int f \, d\mathbf{P}_k \to \int f \, d\mathbf{P}$ for all bounded, uniformly continuous functions f.
- $\mathbf{P}_k(A) \to \mathbf{P}(A)$ for all measurable sets A with $\mathbf{P}(\partial A) = 0$.

We wish to prove that $\mathbf{P}^n_{a_n} \Rightarrow \mathbf{P}_a$ when $a_n \to a$.

Weak Convergence in $D[0, T]$

Definition

A sequence of probability measures $\mathbf{P}_m$ is tight if there for any $\epsilon > 0$ exists a compact set J such that $\mathbf{P}_m(J) > 1 - \epsilon$ for all m.

Theorem

Suppose that $\mathbf{P}_m, \mathbf{P}$ are probability measures on $D[0, T]$ such that

- $\mathbf{P}_m^{t_1,\ldots,t_N} \Rightarrow \mathbf{P}^{t_1,\ldots,t_N}$ *for all* $t_1, \ldots, t_N$ *in* $[0, T]$.
- *The measures $\mathbf{P}_m$ are tight: There are constants $C, a, b, c > 0$ such that for all m and $0 \leq t_1 < t_2 < t_3 \leq T$,*

$$E_{\mathbf{P}_m}(|X_{t_2} - X_{t_1}|^a |X_{t_3} - X_{t_2}|^b) \leq C(t_3 - t_1)^{1+c},$$

then $\mathbf{P}_m \Rightarrow \mathbf{P}$.

Since our estimate for Chentsov's criterion is independent of n, we know that the measures $\mathbf{P}^n_{a_n}$ are tight.

From Billingsley:

Theorem

Let $\mathbf{P}$ be a probability measure, and let $\mathbf{P}_m$ be a sequence of probability measures. Suppose that

- *$\mathcal{A}$ is a π-system*
- *Every open set is a countable union of elements in $\mathcal{A}$.*

If $\mathbf{P}_m(A) \to \mathbf{P}(A)$ for all $A \in \mathcal{A}$, then $\mathbf{P}_m \Rightarrow \mathbf{P}$.

To prove that $\mathbf{P}_{a_n}^{n,t_1,\dots,t_k} \Rightarrow \mathbf{P}_a^{t_1,\dots,t_k}$, we only have to prove that

$$\mathbf{P}_{a_n}^{n,t_1,\dots,t_k}(\omega(t_i) \in B_i) \to \mathbf{P}_a^{t_1,\dots,t_k}(\omega(t_i) \in B_i)$$

for all balls B_i, $1 \le i \le k$.

Theorem

We have that

$$\mathbf{P}_{a_n}^n \Rightarrow \mathbf{P}_a$$

when $a_n \to a$.

Conditional Measure (Brownian Motion)

The conditional measure $\mathbf{P}_{a,b,T}$ is defined in [VSV97] by

$$\mathbf{P}_{a,b,T}(B) = \mathbf{P}_a(B|(\omega(T) = b)).$$

Since we are conditioning on a set of measure 0, it is defined by probability densities. The Chentsov criterion holds [VSV97], so it gives full measure to the Skorokhod paths which start at a and end up at b at time T, and it models a Brownian motion in K going from a to b in time T.

Conditional Measure (Random Walk)

We define the conditional measure $\mathbf{P}^n_{a_n,b_n,T}$ of a Borel set B by

$$\mathbf{P}^n_{a_n,b_n,T}(B) = \frac{\mathbf{P}^n_{a_n}(B \cap (\omega(T) = b_n))}{\mathbf{P}^n_{a_n}(\omega(T) = b_n)}.$$

It gives full measure to the Skorokhod paths which start at a_n and end up at b_n at time T.

We wish to prove that $\mathbf{P}^n_{a_n,b_n,T} \Rightarrow \mathbf{P}_{a,b,T}$ when $a_n \to a$ and $b_n \to b$. The proof for convergence of the finite-dimensional distributions goes exactly as for the unconditioned measures. The difficult part is tightness, and the proof is similar to the one in [DVV94].

Modulus of Continuity

Definition

The "modulus of continuity" for a Skorokhod path ω is

$$m(\omega : \delta) = \sup_{\substack{s_1 < s < s_2 \\ |s_2 - s_1| < \delta}} \min\{|\omega(s_2) - \omega(s)|, |\omega(s) - \omega(s_1)|\}.$$

Theorem

Let $\mathbf{P}_k$ be a sequence of probability measures on $D[0, T]$. TFAE:

- *The sequence of measures $\mathbf{P}_k$ is tight.*
- *For every $\eta > 0$,*

$$\lim_{\delta \to 0} \mathbf{P}_k(\{\omega : m(\omega : \delta) \geq \eta\}) = 0$$

uniformly in k.

Comparison with Unconditioned Measure

Define

$$m_1(\omega : \delta) = \sup_{\substack{s_1 < s < s_2, s_2 - s_1 < \delta \\ s_2 \leq 3T/4}} \min\{|\omega(s_2) - \omega(s)|, |\omega(s) - \omega(s_1)|\}$$

With $A_1 = \{\omega : m_1(\omega : \delta) \geq \eta\}$, we first prove that

$$\lim_{\delta \to 0} \mathbf{P}^n_{a_n, b_n, T}(A_1) = 0$$

uniformly in n. The advantage here is that $3T/4$ is far away from T where we are conditioning.

$$\mathbf{P}^n_{a_n, b_n, T}(A_1) \leq C\mathbf{P}^n_{a_n}(A_1)$$

for some constant C. The measures $\mathbf{P}^n_{a_n}$ are tight so for every $\eta > 0$,

$$\lim_{\delta \to 0} \mathbf{P}^n_{a_n, b_n, T}(A_1) \leq C \lim_{\delta \to 0} \mathbf{P}^n_{a_n}(A_1) = 0$$

uniformly in n.

Time-reversal

We can do the same over the interval $[T/4, T]$ by time-reversal:

$$x^*(s) = x(T - s - 0), \quad 0 \leq s < T$$

and $x^*(T) = x(0)$. The time reversal is a Borel function which is involutive. Define the probability measure $\mathbf{P}^*(E) = \mathbf{P}(E^*)$. With this definition

$$\left(\mathbf{P}^n_{a_n, b_n, T}\right)^* = \mathbf{P}^n_{b_n, a_n, T}$$

This comes from stochastic continuity, which means that if $s_i < s$, then

$$\mathbf{P}^n_{a_n, b_n, T}(\omega : |X_s - X_{s_i}| > \epsilon) \to 0.$$

as $s_i \to s$.

Theorem

We get that $\mathbf{P}^n_{a_n, b_n, T} \Rightarrow \mathbf{P}_{a,b,T}$ when $a_n \to a$ and $b_n \to b$.

Uniform Convergence of the Measures

We have proved that $\mathbf{P}^n_{a_n,b_n,T} \Rightarrow \mathbf{P}_{a,b,T}$ when $a_n \to a$ and $b_n \to b$. The convergence is also uniform when a and b vary in a compact set: If g is any bounded continuous function on $D[0, T]$, then

$$\int_{D[0,T]} g(\omega)\, d\mathbf{P}^n_{a_n,b_n,T}(\omega) \to \int_{D[0,T]} g(\omega)\, d\mathbf{P}_{a,b,T}(\omega)$$

is uniform with respect to a and b varying in a compact set.

Feynman-Kac

$$e^{-tP_n^\alpha}g(x) = e^{-t\mathcal{F}_n^{-1}Q_n^\alpha\mathcal{F}_n}g(x)$$
$$= \mathcal{F}_n^{-1}e^{-tQ_n^\alpha}\mathcal{F}_n g(x) = [\mathcal{F}_n^{-1}e^{-tr_n^\alpha} * g](x) = [\sigma_{n,t} * g](x),$$

$$e^{-(t/N)Q_n}e^{-(t/N)V_n}g(x) = \int_{G_n} \sigma_{n,t/N}(y-x)e^{-(t/N)v_n(y)}g(y)\,d\mu_n(y)$$

$$\left(e^{-(t/N)Q_n}e^{-(t/N)V_n}\right)^N g(x)$$
$$= \int_{G_n^N} \sigma_{n,t/N}(x - x_1)\cdots\sigma_{n,t/N}(x_{N-1} - x_N)$$
$$\cdot\, e^{-(t/N)\sum_{i=1}^N v_n(x_i)}g(x_N)\,d\mu_n(x_1)\cdots d\mu_n(x_N)$$

By using the Trotter product formula one obtains Feynman-Kac.

$$e^{-tH_n}(j_n, k_n) = \int_{D[0,t]} e^{-\int_0^t v_n(\omega(s))\,ds}\,d\mathbf{P}^n_{j_n,k_n,t}(\omega)\cdot\sigma_{n,t}(k_n - j_n)q^{-n}.$$