Quantization causes waves

Smooth finitely computable functions are affine

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1. Experiments and problems

2. Representations of automata maps in real and complex spaces
   - General automata
   - Finite automata

3. Relations to quantum theory
   - Physical measurements, information, and quantum theory
   - Torus windings and wave functions
The first theme

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2. Representations of automata maps in real and complex spaces
   - General automata
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   - Physical measurements, information, and quantum theory
   - Torus windings and wave functions
Notion: *system* (R. E. Kalman, 1969)

A (discrete) system (or, a system with a discrete time $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$) is a 5-tuple $\mathcal{A} = \langle I, S, \emptyset, S, O \rangle$ where
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- $\mathcal{S}: \mathcal{I} \times \mathcal{S} \rightarrow \mathcal{S}$ is a state transition function;
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During the talk, an **automaton** $\mathcal{A}(s_0)$ (or, a *transducer*) is a system where one of the states, $s_0 \in S$, is fixed; it is called the initial state.
Experiments and problems

Systems, automata and word maps

\[ s_{i+1} = S(\chi_i, s_i) \]

\[ \xi_i = O(\chi_i, s_i) \]

Input: \( \cdots \chi_{i+1}\chi_i \)

State transition diagram:

Output: \( \xi_i\xi_{i-1}\cdots\xi_0 \)

Figure: Schematics of an automaton

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Figure: Example state diagram of automaton ($I = \varnothing = \{0, 1\}; \#S = 5, s_0 = 1$)
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Therefore given an automaton $A$ with $p$-letter input/output alphabets $I = O = \mathbb{F}_p = \{0, 1, \ldots, p - 1\}$ ($p$ a prime), under a natural one-to-one correspondence between the set of all left-infinite words and the space of $p$-adic integers $\mathbb{Z}_p$, automaton function $f_A : \mathbb{Z}_p \to \mathbb{Z}_p$ is a 1-Lipschitz map w.r.t. $p$-adic metric on $\mathbb{Z}_p$. Vice versa, given a 1-Lipschitz map $f : \mathbb{Z}_p \to \mathbb{Z}_p$, the map $f$ is an automaton function for a suitable automaton $A$. 

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Automaton function \( f_A : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) is a 1-Lipschitz map w.r.t. \( p \)-adic metric on \( \mathbb{Z}_p \). Vice versa, given a 1-Lipschitz map \( f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \), the map \( f \) is an automaton function for a suitable automaton \( A \).

Each letter of output word depends \textit{only} on those letters of input word which have already been fed to the automaton. Letters of the input word can naturally be ascribed to ‘causes’ while letters of corresponding output word can be regarded as ‘effects’. ‘Causality’ just means that effects depend only on causes which ‘already have happened’. Therefore \textbf{automata serve as adequate mathematical formalism for causality principle}.
Given an automaton $\mathcal{A}$ with a $p$-letter input/output alphabets $\mathbb{F}_p = \{0, 1, \ldots, p - 1\}$, let $f = f_{\mathcal{A}}$ be the automaton function. For $k = 1, 2, \ldots$ let $E_k(f_{\mathcal{A}})$ be the set of all points of the Euclidean unit square $I^2 = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ s.t.:

$$E_k(f_{\mathcal{A}}) = \left\{ \left( \frac{x \mod p^k}{p^k}; \frac{f(x) \mod p^k}{p^k} \right) : x \in \mathbb{Z}_p \right\}$$

where given $z = \sum_{j=0}^{\infty} \chi_j p^j \in \mathbb{Z}_p$, we denote $z \mod p^k$ a non-negative integer whose base-$p$ expansion is $\chi_{k-1} p^{k-1} + \cdots + \chi_i p + \chi_0$.

This way we define the automaton function on numbers $n/p^k \in [0, 1]$.

Low order digits are feeded to/outputted from the automaton prior to higher order digits!
Experimenting with automata

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Experimenting with automata

\[ w = \chi_{k-1} \cdots \chi_1 \chi_0 \quad \xi_{k-1} \cdots \xi_1 \xi_0 = f_{\mathcal{A}}(w) \]

\[ (0.\chi_{k-1} \cdots \chi_1 \chi_0; 0.\xi_{k-1} \cdots \xi_1 \xi_0) \]

\[ E_k(f_{\mathcal{A}}) = \{(0.w; 0.f(w)) : w \text{ runs over words of length } k\} \]

Experimentally it can be observed that \( E_k(f_{\mathcal{A}}) \) when \( k \rightarrow \infty \) basically exhibits behaviour of two kinds only:
Experiments and problems

Experimenting with automata

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1. \( E_k(f) \) is getting more and more dense so that at \( k \to \infty \) they fill the unit square completely
Experimenting with automata

\[ w = \chi_{k-1} \cdots \chi_1 \chi_0 \quad \xi_{k-1} \cdots \xi_1 \xi_0 = f_\mathcal{A}(w) \]

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2. \( E_k(f) \) is getting less and less dense and with pronounced straight lines that constitute windings of a torus
Experimenting with automata

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The goal of the talk is to explain the following:

- what really happens (=mathematical results), and
Experimenting with automata

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The goal of the talk is to explain the following:

- what really happens (=mathematical results), and
- how the results could be related to quantum theory interpretation.
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2. Representations of automata maps in real and complex spaces
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The next topic:

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The automata 0-1 law

Denote via $\alpha(f_A)$ Lebesgue measure of the plot of $A$, i.e., of the closure $P(f_A) = P(A)$ of the union of sets $E_k(f_A)$, $k = 1, 2, 3, \ldots$

Theorem (The automata 0-1 law; V. A., 2009)

*Given an automaton function $f = f_A$, either $\alpha(f) = 0$, or $\alpha(f) = 1$.***
Representations of automata maps in real and complex spaces

**The automata 0-1 law**

Denote via $\alpha(f_{\mathcal{A}})$ Lebesgue measure of the plot of $\mathcal{A}$, i.e., of the closure $P(f_{\mathcal{A}}) = P(\mathcal{A})$ of the union of sets $E_k(f_{\mathcal{A}})$, $k = 1, 2, 3, \ldots$.

**Theorem (The automata 0-1 law; V. A., 2009)**

*Given an automaton function $f = f_{\mathcal{A}}$, either $\alpha(f) = 0$, or $\alpha(f) = 1$.***

These alternatives correspond to the cases $P(f)$ is nowhere dense in $\mathbb{I}^2$ and $P(f) = \mathbb{I}^2$, respectively.
Representations of automata maps in real and complex spaces

General automata

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Denote via $\alpha(f_\mathcal{A})$ Lebesgue measure of the plot of $\mathcal{A}$, i.e., of the closure $\mathcal{P}(f_\mathcal{A}) = \mathcal{P}(\mathcal{A})$ of the union of sets $E_k(f_\mathcal{A})$, $k = 1, 2, 3, \ldots$.

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These alternatives correspond to the cases $\mathcal{P}(f)$ is nowhere dense in $\mathbb{I}^2$ and $\mathcal{P}(f) = \mathbb{I}^2$, respectively.

We will say for short that an automaton $\mathcal{A}$ is of measure 1 if $\alpha(f_\mathcal{A}) = 1$, and of measure 0 if otherwise.

Finite automata (=automata with a finite number of states) are all of measure 0

Therefore if $\mathcal{A}$ is a finite automaton then $\mathcal{P}(f_\mathcal{A})$ is nowhere dense in $\mathbb{I}^2$ and thus $\mathcal{P}(f_\mathcal{A})$ cannot contain ‘figures’, but it may contain ‘lines’.
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Plots of linear finite automata

Given an automaton function is $f_\mathcal{A}$, denote via $\text{AP}(\mathcal{A}) = \text{AP}(f_\mathcal{A})$ the set of all accumulation points of the plot $\text{P}(\mathcal{A}) \subset \mathbb{T}^2$ on the torus $\mathbb{T}^2$. 
Plots of linear finite automata

Given an automaton function is \( f_\mathcal{A} \), denote via \( \text{AP}(\mathcal{A}) = \text{AP}(f_\mathcal{A}) \) the set of all accumulation points of the plot \( \mathcal{P}(\mathcal{A}) \subset \mathbb{T}^2 \) on the torus \( \mathbb{T}^2 \).

Important example: linear automata

Let \( \mathcal{A} \) be a finite automaton, and let \( f_\mathcal{A}(z) = f(z) = az + b \) (\( a, b \in \mathbb{Z}_p \cap \mathbb{Q} \) then). Considering \( \mathbb{I}^2 \) as a surface of the torus \( \mathbb{T}^2 \), we have that

\[
\text{AP}(f) = \{(x \mod 1; (ax + b) \mod 1) \in \mathbb{T}^2 : x \in \mathbb{R}\}
\]

is a link of \( N_f \) torus knots either of which is a cable (winding) with slope \( a \) of the unit torus \( \mathbb{T}^2 \): If \( a = q/k, b = r/s \) are irreducible fractions, \( d = \gcd(k, s) \) then \( N_f \) is multiplicative order of \( p \) modulo \( s/d \). Each cable winds \( q \) times around the interior of \( \mathbb{T}^2 \) and \( k \) times around \( Z \)-axis.

Given \( a, b \in \mathbb{Z}_p \), the mapping \( z \mapsto az + b \) is an automaton function of a suitable finite automaton if and only if \( a, b \in \mathbb{Z}_p \cap \mathbb{Q} \).
Plots of linear finite automata

Important example: linear automata

Let $A$ be a finite automaton, and let $f_A(z) = f(z) = az + b \ (a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ then). Considering $\mathbb{I}^2$ as a surface of the torus $\mathbb{T}^2$, we have that

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is a link of $N_f$ torus knots either of which is a cable (=winding) with slope $a$ of the unit torus $\mathbb{T}^2$: If $a = q/k, b = r/s$ are irreducible fractions, $d = \gcd(k, s)$ then $N_f$ is multiplicative order of $p$ modulo $s/d$. By using cylindrical coordinates ($\mathbb{T}^2: (r_0 - R)^2 + z^2 = A^2; R = A = 1$ for unit torus) we get:

$$\begin{bmatrix} r_0 \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} R + A \cdot \cos (ax - 2\pi b \cdot p^\ell) \\ x \\ A \cdot \sin (ax - 2\pi b \cdot p^\ell) \end{bmatrix}, \ x \in \mathbb{R}, \ell = 0, 1, 2, \ldots$$
Important example: linear automata

Let $\mathcal{A}$ be a finite automaton, and let $f_{\mathcal{A}}(z) = f(z) = az + b$ ($a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ then). Considering $\mathbb{I}^2$ as a surface of the torus $\mathbb{T}^2$, we have that

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is a link of $N_f$ torus knots either of which is a cable (=winding) with slope $a$ of the unit torus $\mathbb{T}^2$: If $a = q/k, b = r/s$ are irreducible fractions, $d = \gcd(k, s)$ then $N_f$ is multiplicative order of $p$ modulo $s/d$. Therefore the plot of $f (=of the automaton $\mathcal{A}$) can be described by $N_f$ complex-valued functions:

$$\text{AP}(\mathcal{A}) = \text{AP}(az + b) \leftrightarrow e^{i(ax - 2\pi b \cdot p^\ell)}; \ (x \in \mathbb{R}, \ell \in \mathbb{N}_0)$$
Q: What *smooth* curves are finitely computable; i.e. what *smooth* curves may lie in the plot $\mathcal{P}(\mathcal{A})$ of a finite automaton $\mathcal{A}$?
The affinity of smooth finitely computable functions

- **Q:** What *smooth* curves are finitely computable; i.e. what *smooth* curves may lie in the plot $P(\mathcal{A})$ of a finite automaton $\mathcal{A}$?

- **A:** Only straight lines (=cables of torus with rational $p$-adic slopes and rational $p$-adic constant terms).
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Let $g$ be a two times differentiable function (w.r.t. the metric in $\mathbb{R}$) defined on $[\alpha, \beta] \subset [0, 1)$ and valuated in $[0, 1)$; let $g''$ be continuous on $[\alpha, \beta]$. If $(x; g(x)) \in P(\mathcal{A})$ for all $x \in [\alpha, \beta]$ then there exist $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ such that $g(x) = ax + b$ for all $x \in [\alpha, \beta]$ and $AP(az + b) \subset P(\mathcal{A})$. Given a finite automaton $\mathcal{A}$, there are no more than a finite number of $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ such that $AP(az + b) \subset P(\mathcal{A})$. 
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Let $g$ be a two times differentiable function (w.r.t. the metric in $\mathbb{R}$) defined on $[\alpha, \beta] \subset [0, 1)$ and valuated in $[0, 1)$; let $g''$ be continuous on $[\alpha, \beta]$. If $(x; g(x)) \in P(\mathcal{A})$ for all $x \in [\alpha, \beta]$ then there exist $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ such that $g(x) = ax + b$ for all $x \in [\alpha, \beta]$ and $AP(az + b) \subset P(\mathcal{A})$. Given a finite automaton $\mathcal{A}$, there are no more than a finite number of $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ such that $AP(az + b) \subset P(\mathcal{A})$.

Actually this means that smooth curves in the plot of a finite automaton constitute a finite union of torus links, and every link consists of a finite number of torus knots with the same slope.

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The theorem can be considered as a contribution to the theory of computable functions since the main result means that a finite automaton can compute only a very restricted class of smooth functions; namely, only affine ones.
The affinity of smooth finitely computable functions


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The theorem holds for automata with $m$ inputs and $n$ outputs: Smooth surfaces in the plot (in multidimensional torus) constitute a finite number of families of multidimensional torus windings, and each family is a finite collection of windings with the same matrix $A$.

\[
\mathbf{AP}(zA + b) \iff e^{i(xA - 2\pi b \cdot \ell)}; \quad (x \in \mathbb{R}^m; \ b \in \mathbb{R}^n; \ \ell \in \mathbb{N}_0)
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The affinity of smooth finitely computable functions


Let $g$ be a two times differentiable function (w.r.t. the metric in $\mathbb{R}$) defined on $[\alpha, \beta] \subset [0, 1)$ and valuated in $[0, 1)$; let $g''$ be continuous on $[\alpha, \beta]$. If $(x; g(x)) \in P(\mathcal{A})$ for all $x \in [\alpha, \beta]$ then there exist $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ such that $g(x) = ax + b$ for all $x \in [\alpha, \beta]$ and $AP(az + b) \subset P(\mathcal{A})$. Given a finite automaton $\mathcal{A}$, there are no more than a finite number of $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ such that $AP(az + b) \subset P(\mathcal{A})$.

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Thus the theorem can be applied to study hash functions since they all are automata functions of finite automata with multiple inputs/outputs.
Final part

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On physical measurements

- A **physical law** is a mathematical correspondence between quantities of impacts a physical system is exposed to and quantities of responses the system exhibits.
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On physical measurements

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- An experimental curve is a smooth curve (the $C^2$-smoothness is common) which is the best approximation of the set of the experimental points.
- A physical law is a curve which can be approximated by the experimental curves with the highest achievable accuracy.
On physical measurements

Let physical quantities which correspond to impacts and reactions are quantized; i.e., take only values, say, $0, 1, \ldots, p - 1$. Then, once the system is exposed to a sequence of $k$ impacts, it outputs corresponding sequence of $k$ reactions. Considering these sequences as base-$p$ expansions of natural numbers $z = \alpha_{k-1}p^{r-1} + \cdots + \alpha_0$, after the normalization $\frac{z}{p^k}$ we obtain experimental points in a unit square of $\mathbb{R}^2$. 
On physical measurements

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Our main theorem shows that if the number of states of automaton which corresponds to a physical system is much less than the length of input sequence of impacts then experimental curves necessarily tend to straight lines (or torus windings, under a natural map of the unit square onto a torus). This implies the linearity of a physical law.
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Q: Once Prof. A. Yu. Khrennikov asked a question: “Why mathematical formalism of quantum theory is linear?”
A: Quantization + Small set of states $\Rightarrow$ Linearity
John Archibald Wheeler (July 9, 1911 – April 13, 2008) was a prominent American physicist known for his contribution to general relativity and quantum theory. In 1990, Wheeler suggested that information is fundamental to the physics of the universe and started developing the informational interpretation of physics. He wrote:
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‘It from bit’ symbolizes the idea that every item of the physical world has at bottom — a very deep bottom, in most instances — an immaterial source and explanation; that which we call reality arises in the last analysis from the posing of yes-no questions and the registering of equipment-evoked responses; in short, that all things physical are information-theoretic in origin and that this is a participatory universe.
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Information in physics: Wheeler’s ‘it from bit’ doctrine

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Let’s give some mathematical reasoning why ‘it’ is indeed ‘from bit’.
The next topic:

1. Experiments and problems

2. Representations of automata maps in real and complex spaces
   - General automata
   - Finite automata

3. Relations to quantum theory
   - Physical measurements, information, and quantum theory
   - Torus windings and wave functions
Accumulation points of a plot of a finite linear automaton (whose automaton function is then $f : z \mapsto az + b$ for suitable $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$) look like a finite collection of waves with the same wavenumber $a$ (up to a normalization s.t. $\hbar = 1$), where $x$ stands for position, $2\pi b$ for angular frequency $\omega$ and $p^\ell$ for time $t$.

$$\text{AP}(\mathcal{A}) = \text{AP}(az + b) \longleftrightarrow e^{i(ax - 2\pi b \cdot p^\ell)}; \quad (x \in \mathbb{R}, \ell \in \mathbb{N}_0)$$

$$\Psi = Ae^{i(px - \omega t)}$$
Speculation: Automata as models of quantum systems

Accumulation points of a plot of a finite linear automaton (whose automaton function is then \( f : z \mapsto az + b \) for suitable \( a, b \in \mathbb{Z}_p \cap \mathbb{Q} \)) look like a finite collection of waves with the same wavenumber \( a \) (up to a normalization s.t. \( \hbar = 1 \)), where \( x \) stands for position, \( 2\pi b \) for angular frequency \( \omega \) and \( p^\ell \) for time \( t \). Consider a special case when \( a \in \mathbb{Z} \) and \( b = 0 \):

\[
\text{AP}(\mathcal{A}) = \text{AP}(az) \iff e^{i(ax)}; \quad (x \in \mathbb{R}, \ell \in \mathbb{N}_0)
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\[ \text{AP}(\mathcal{A}) = \text{AP}(az) \leftrightarrow e^{i(ax)}; \quad (x \in \mathbb{R}, \ell \in \mathbb{N}_0) \]

It is reasonable to suggest that indeed \( a \) is the wavenumber (cf. left and right pictures); thus wavelength \( \lambda = \frac{1}{a} \). Moreover, then \( \omega = \lambda^{-1} = a \); and note that \( \text{AP}(az + at) = \text{AP}(az) \) for every \( t \in \mathbb{Z} \).
Speculation: Automata as models of quantum systems

$\text{AP}(\mathcal{A}) = \text{AP}(az + b) \leftrightarrow e^{i(ax - 2\pi b \cdot p^\ell)}; \quad (x \in \mathbb{R}, \ell \in \mathbb{N}_0)$

The “time-looking” multiplier $p^\ell$ is a proper time of the automaton.
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The “time-looking” multiplier \( p^\ell \) is a proper time of the automaton. Namely, multiplying by \( p \) corresponds to on step of the automaton: \((p^\ell x) \mod 1\) is an \( \ell \)-step shift of base-\( p \) expansion of \( x \in \mathbb{R} \).
Speculation: Automata as models of quantum systems

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The “time-looking” multiplier \( p^\ell \) is a proper time of the automaton. Can \( p^\ell \) be treated a physical time?
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The “time-looking” multiplier $p^\ell$ is a proper time of the automaton. Can $p^\ell$ be treated a physical time? Yes!

Just take $p$ close to 1 (forgetting for a moment that $p$ is a base of a radix system); i.e., $p = 1 + \tau$ where $\tau$ is small.

For instance, assume that $\tau$ is a Planck time (=a quant of time) or other time interval which is less then the accuracy of measurements and thus can not be measured.
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Then $p^\ell \approx 1 + \ell\tau$ and therefore for large $\ell$ we see that $\ell\tau = t$ is just a time. And here we are:

$$e^{i(ax-2\pi b\cdot p^\ell)} \approx e^{i(ax-2\pi b\cdot(1+t))} = c \cdot e^{i(ax-2\pi b\cdot t)}$$

← the wave!!!
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Is it mathematically correct to put \( p = 1 + \tau \) where \( 0 < \tau \ll 1 \)?
Speculation: Automata as models of quantum systems

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Is it mathematically correct to put \( p = 1 + \tau \) where \( 0 < \tau \ll 1 \)? Yes, if one uses \( \beta \)-expansions (introduced by Rényi—Parry in 1957–1960) rather than base \( p \)-expansions.
Using $\beta$-expansions of numbers

To construct a plot of an automaton over a $p$-letter alphabet we use base-$p$ expansions of numbers: To every pair of input/output words

$$\text{input word } \chi_{k-1} \cdots \chi_1 \chi_0 \rightarrow \text{output word } \xi_{k-1} \cdots \xi_1 \xi_0$$

we put into the correspondence the point on the torus

$$\left(\chi_{k-1}p^{-1} + \cdots + \chi_1p^{-k+1} + \chi_0p^{-k}; \xi_{k-1}p^{-1} + \cdots + \xi_1p^{-k+1} + \xi_0p^{-k}\right)$$
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We may take $\beta \in \mathbb{R}^+$ such that $\lfloor \beta \rfloor = p - 1$ and use $\beta$-expansions rather than base-$p$ expansions; that is, we put into the correspondence to the pair of input/output words the following point on the torus:

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Vladimir Anashin (MSU-RAS)
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We may take $\beta \in \mathbb{R}^+$ such that $[\beta] = p - 1$ and use $\beta$-expansions rather than base-$p$ expansions; that is, we put into the correspondence to the pair of input/output words the following point on the torus:

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- necessarily the input/output alphabets are binary, and
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Then

- necessarily the input/output alphabets are binary, and
- any torus link will be dense and can be ascribed to a matter wave $ce^{i(ax-2\pi bt)}$ where $x$, $t$ are space and time coordinates accordingly.
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Then (as $\beta = 1 + \tau$ with $\tau$ small)

- necessarily the input/output alphabets are binary, and
- any torus link can be ascribed to a matter wave $ce^{i(ax - 2\pi bt)}$ where $x, t$ are space and time coordinates accordingly.

Therefore our main theorem can serve a mathematical reasoning why a specific ‘it’ — the matter wave, which is a core of quantum theory — is indeed ‘from bit’; that is, from sufficiently long binary inputs of an automaton with a relatively small number of states.
Using $\beta$-expansions of numbers

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The approach seemingly reveals more correspondences between physical entities and mathematical properties of automata, for instance:

- helicity corresponds to the sign of $a$;
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- probability of finding a particle at a certain point of space corresponds to the average amplitude when \( t \in [0, \infty) \);
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- helicity corresponds to the sign of $a$;
- probability of finding a particle at a certain point of space corresponds to the average amplitude when $t \in [0, \infty)$;
- automata with multiple inputs/outputs correspond to finite-dimensional Hilbert spaces (to include into the consideration infinite-dimensional Hilbert spaces one needs to consider automata of measure 0 rather than just finite ones);
Using $\beta$-expansions of numbers

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- automata with multiple inputs/outputs correspond to Hilbert spaces;
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pure states correspond to **ergodic** linear subautomata, mixed states correspond to automata states leading to more than 1 ergodic subautomata; entangled states correspond to states from ergodic subautomata of automata with multiple inputs/outputs.
Using $\beta$-expansions of numbers

Interpretation of the case $\beta = 1 + \tau$ where $0 < \tau \ll 1$

In our model, $\beta^j$ may be interpreted as a time which is needed to acquire the next $j$-th bit; so the time $T_k$ needed to acquire a $k$-bit information is exponential in $k$, namely $T_k = \frac{1}{\tau} (\beta^k - 1)$. 

Vladimir Anashin (MSU-RAS)

Quantization causes waves

September 8, 2015, Belgrade
Using $\beta$-expansions of numbers

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- Note that then $T_k \approx k$ for large $k$. 
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- But E. Lerner recently has shown that in the latter case the plot will be a polygon ("body") rather than a torus winding ("wave").
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So in QT acquiring of information actually is exponential in time, in a contrast to classical case when this is linear. In our model, classical case appears as a limit case at $\tau \to 0$; i.e., as it should be: In a contrast to QT, there are arbitrarily small time intervals in classical case.
Using $\beta$-expansions of numbers

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Messages of the talk

Discreteness+Causality+Finiteness $\Rightarrow$ Waves
Messages of the talk

- Discreteness + Causality + Finiteness $\Rightarrow$ Waves
- Waves, the *its*, are indeed from *bits*
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- Waves, the *its*, are indeed from *bits*
- Acquisition of information in QT during a measurement is exponential in time (though base is close to 1)
Thank you!
\[ p = 2: f(x) = 1 + x + 4x^2; \]

\[ \alpha(f) = 1. \]
\( f(z) = \frac{11}{15}z + \frac{1}{21}, \quad p = 2. \) (Therefore \( N_f = \text{mult}_7 2 = 3 \))
$f_1(z) = -2z + \frac{1}{3}; f_2(z) = \frac{3}{5}z + \frac{2}{7}, (p = 2)$. 
\[ f_1(z) = -2z + 1; \]
\[ f_2(z) = 3z + 2. \]
Quantization causes waves
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