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STABILITY OF THE RELATIVE EQUILIBRIA IN THE GENERALIZED J_2 PROBLEM

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SUMMARY: For a large class of concrete astronomical situations, the motion of celestial bodies is modelled by dynamical systems associated to a potential function $\alpha/r + \varepsilon U$ (r = distance between particles, α = real constant, ε = real small parameter, U = perturbing potential). In this paper the nonlinear stability of the relative equilibrium orbits corresponding to such a potential is being investigated using a less usual method, which combines a block diagonalization technique with the reduction procedure. The test points out certain nonlinearly stable orbits, and is inconclusive for the remaining equilibria. The latter ones are treated via linearization; all of them prove instability. The nonlinearly stable orbits remain stable under any perturbation that preserves the conserved momentum.

1. INTRODUCTION

The orbital stability of heavenly bodies always constituted one of the main problems of astronomy. Starting from the natural question: is our planetary system stable or not, then extending it to other bodies of the solar system, a huge number of outstanding scientists dedicated their efforts to contribute to the understanding and solution of such a tremendous problem. Some landmark names are sufficient: Lagrange, Laplace, Poisson, Maxwell, Haret, Poincaré, Liapunov, Birkhoff, Kolmogorov, Arnold, Moser (for a more complete list see, e.g., Pal 1991; Diacu and Holmes 1996). To study this problem, the most various methods and techniques were used, belonging mainly to celestial mechanics and - much more recently - to the more general theory of dynamical systems.

The actual astronomical investigations on orbital stability focus on natural and artificial planetary satellites, planetary rings, asteroids, comets, stellar satellites (from planets to particles in surrounding disks), binary stars of all kinds, accretion disks around AGNs, stellar clusters, clusters of galaxies, and so forth. The class of astronomical situations that involve the study of dynamical stability is mighty large. In such studies, one of the most used models of force field is that featured by a potential function $\alpha/r+\varepsilon U$, where α is a real constant, ε is a small parameter, U is the perturbing potential, while r stands for the distance between two particle in this field. In other words, if $\alpha > 0$ the associated two-body problem is equivalent to the perturbed Kepler problem in which the perturbation depends on a small parameter.

In this paper we shall restrict the class of perturbing potentials to the (still very large) range that leads to the so-called J_2 problem (see below). At the same time we relax the condition of perturbed Keplerian motion by letting the parameter α run all along the real line. Finally, we deal with a single aspect (but very important) of the problem: the relative equilibria.

To be more explicit, let us consider the gravitational field generated by a celestial body that presents geometric and dynamical symmetry with respect to an axis. In this case the potential is expandable in spherical functions, i.e. it can be expressed by the sum of the Newtonian potential and the zonal harmonics. The study of the motion in such a field, when only the second zonal harmonic is taken into account as perturbing factor, bears the name of J_2 *problem*. It is also called the *main problem of space dynamics* (e.g. Saari 1974; Belenkii 1981) because such a model was first used to investigate the artificial satellite motion around the Earth approximated by a rotation ellipsoid (the deviation from a sphere being featured only by J_2). If the motion of the satellite body is confined to the equatorial plane (the only case which admits equilibria), then the model reduces to the two-body problem associated to a potential function of the form $\alpha/r + \beta/r^3$, with r = distance between bodies, and α, β = positive constants.

Considering the more general case of the J_2 problem in which the Newtonian-type force may be attracting/zero/repelling (as α is positive/zero/negative, respectively), whereas the perturbative term that contains J_2 may be positive/zero/negative, almost all equilibria are found to lie in the equatorial plane of the field-generating body. (There also is a degenerate case: $\alpha = 0$, $J_2 = 0$, i.e. the force-free field, which excepts from this situation.) So, we define the *generalized J_2 problem*: the "equatorial" J_2 problem with $\alpha, \beta \in \mathbf{R}$, $\alpha^2 + \beta^2 \neq 0$, to which is added (for completeness) the degenerate case of the force-free field.

This model corresponds to a wide range of concrete astronomical situations. To give some examples, the motion in the equatorial plane of an oblate, rotating star is such a situation (here α can be positive/zero/negative as the Newtonian gravitational attraction is stronger than/equal to/weaker than the repelling radiative force). If we consider that the central body is a spherical or ellipsoidal planet which also acts on the satellite body through the pressure of the diffusely re-emitted radiation of the "landlord" star, the mathematical formalism is identical. The motion around bodies that generate Schwarzschild-type fields (radiation included or not) also join this model (e.g. Stoica and Mioc 1997; Mioc and Stavinschi 1998). Of course, particular cases of the Coulombian field (e.g. Sommerfeld 1951; Belenkii 1981) or well-known models, as the Newtonian field, the purely radiative field, the classical photogravitational field, or the already mentioned force-free field, are recovered, too.

To study the nonlinear stability of steadily rotating configurations of bodies (relative equilibria), a very efficient tool is provided by a combination of the block-diagonalization method proposed by Marsden *et al.* (1989), and developed by Maddocks (1991) and Simó *et al.* (1991), with the classical reduction procedure. This technique, particularized by Zombro and Holmes (1993) to systems with a finite number of degrees of freedom and with a single rotational symmetry, was applied for instance to the study of the relative equilibrium configurations in the $(n + 1)$-body problem; cf. Elmabsout (1988, 1990, 1994, 1996) and the references therein.

We use this technique to reduce the Hamiltonian of the problem with a single cyclic coordinate θ, restricting it in this way to the level set p_θ = constant, where p_θ is the (conserved) momentum conjugate to θ. The relative equilibria of the corresponding amended potential lie in the equatorial plane of the field-generating body, except the above mentioned degenerate situation.

Tackling the Liapunov nonlinear stability of these equilibria, we find that *the nonlinearly stable cases remain stable for the whole class of perturbations that do not affect the conserved momentum.* This means that the stable character of the equatorial circular orbits is preserved under the influence of certain perturbations as, for instance, all other zonal harmonics in the potential expansion, or the diffusely re-emitted radiation pressure.

We also point out situations in which the nonlinear stability test is inconclusive. Applying the classical linearization method to these cases, we obtain that *all corresponding equilibrium orbits (circular or rest points) are linearly unstable.*

2. AMENDED POTENTIAL

We fix the origin of the coordinates in the mass center of the field-generating body, and study the relative motion with respect to it. The corresponding equations of motion are

$$\dot{\mathbf{q}} = \frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial \mathbf{p}}, \qquad (1)$$
$$\dot{\mathbf{p}} = -\frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}},$$

where $\mathbf{q} = (q_1, q_2, q_3) \in \mathbf{R}^3 \setminus \{(0,0,0)\}$ is the configuration vector, while $\mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}^3$ stands for the momentum vector. The class of Hamiltonians we deal with has the form (in suitably chosen units):

$$H(\mathbf{q}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2} - \frac{\alpha}{|\mathbf{q}|} - \varepsilon U\left(\sqrt{q_1^2 + q_2^2}, q_3\right), \qquad (2)$$

in which $\alpha, \varepsilon \in \mathbf{R}$, ε being a small parameter. It is clear that the first two terms in the right-hand side of (2) describe the unperturbed problem, whereas the third term represents the perturbation.

We shall pass to cylindrical coordinates (r, θ, z) and corresponding momenta (p_r, p_θ, p_z) via the transformations

$$(q_1, q_2, q_3) = (r\cos\theta,\ r\sin\theta,\ z), \qquad (3)$$
$$(p_1, p_2, p_3) = (p_r\cos\theta - \frac{p_\theta}{r}\sin\theta,\ p_r\sin\theta + \frac{p_\theta}{r}\cos\theta,\ p_z).$$

Let us observe that, under this change of variables, the motion equations keep their canonical character (however, in the general case the canonical character is lost). By virtue of (3), the Hamiltonian (2) becomes

$$H = \frac{1}{2}\left(p_r^2 + p_z^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\alpha}{\sqrt{r^2+z^2}} - \varepsilon U(r,z), \quad (4)$$

where we kept by abuse the same notation for the Hamiltonian, as well as for the perturbing potential as functions of the new variables introduced through the transformations (3).

Notice that H does not depend on θ. This means that the momentum conjugate to θ is conserved. Therefore we may apply the reduction, confining the Hamiltonian to the level set $L := p_\theta =$ constant.

According to Zombro and Holmes (1993), the amended potential of (2) will have the expression

$$U_L = \frac{L^2}{2r^2} - \frac{\alpha}{\sqrt{r^2+z^2}} - \varepsilon U(r,z). \quad (5)$$

Considering that the perturbing potential is expandable in spherical functions, we can write (e.g. Brouwer and Clemence 1961)

$$\varepsilon U(r,z) = \sum_{n\geq 2} \frac{J_n}{\left(\sqrt{r^2+z^2}\right)^{n+1}} P_n\left(\frac{z}{\sqrt{r^2+z^2}}\right), \quad (6)$$

where J_n stands for the coefficient of the n-th zonal harmonic, while P_n denotes the n-th order Legendre polynomial. For $n = 2$, and denoting $J := J_2$, (5) becomes

$$V := U_L(n=2) = \\ = \frac{L^2}{2r^2} - \frac{\alpha}{\sqrt{r^2+z^2}} - \frac{J}{2(r^2+z^2)^{3/2}}\left[\frac{3z^2}{r^2+z^2} - 1\right], \quad (7)$$

that corresponds to the generalized J_2 problem (in which α and J may take any real value). Observe that the equatorial case ($z = 0$) is equivalent to the so-called Schwarzschild problem (see Stoica and Mioc 1997).

3. RELATIVE EQUILIBRIA

Let us write the equations of motion in an explicit form. By virtue of (4), and taking into account the reduction operated in the previous section, these equations read

$$\begin{aligned} \dot r &= p_r, \\ \dot z &= p_z, \\ \dot p_r &= \frac{p_\theta^2}{r^3} - \frac{\alpha r}{(r^2+z^2)^{3/2}} + \frac{3Jr(r^2-4z^2)}{2(r^2+z^2)^{7/2}}, \\ \dot p_z &= -\frac{\alpha z}{(r^2+z^2)^{3/2}} + \frac{3Jz(3r^2-2z^2)}{2(r^2+z^2)^{7/2}}. \end{aligned} \quad (8)$$

The *relative equilibria* (steadily rotating states) correspond to $\dot\theta =$ constant $(\neq 0)$, $p_\theta = L$ $(\neq 0)$, and $\dot r = 0 = \dot z$, $\dot p_r = 0 = \dot p_z$. The *relative rest equilibria* (also called *nonrotating relative equilibria*) require the same conditions, but with $\dot\theta = 0$, $L = 0$.

Proposition 3.1. *The only situation that admits equilibria for $z \neq 0$ is the degenerate case of the force-free field ($\alpha = 0$, $J = 0$). The corresponding critical point is a relative rest equilibrium.*

Proof. It is clear that for $z =$ constant $(\neq 0)$ rotating relative equilibria cannot exist. Let us suppose that there exist relative rest equilibria for $z = z_e \neq 0$ and $r = r_e > 0$. Then, putting in (8) $p_\theta = L = 0$, multiplying the third equation by $-z$, the fourth equation by r, and adding the resulting expressions (each being equal to zero) together, one gets $Jz_e = 0$, which means either $z_e = 0$, or $J = 0$. The first case leads to a contradiction (we have supposed $z_e \neq 0$). In the latter situation, the last two equations (8) imply $\alpha = 0$. But $\alpha = 0$, $J = 0$ mean force-free field.□

Proposition 3.2. *For $z = 0$ (equatorial plane), the relative equilibria are given by*

$$\alpha r^2 - L^2 r - \frac{3J}{2} = 0. \quad (9)$$

Proof. It is clear from the fourth equation (8) that $\dot p_z = 0$ for $z = 0$. Putting $z = 0$ in the third equation (8) with $\dot p_r = 0$, formula (9) results immediately.□

Now we are in the position to state

Proposition 3.3. *The only relative (or relative rest) equilibria of the "equatorial" generalized J_2 problem have the following characteristics:*

(i) $\alpha > 0$, $J > 0$, $L \in \mathbf{R}$:

$$r_e = \frac{L^2 + \sqrt{L^4 + 6\alpha J}}{2\alpha};$$

(ii) $\alpha > 0$, $J = 0$, $L \in \mathbf{R}$:

$$r_e = \frac{L^2}{\alpha};$$

(iii) $\alpha > 0$, $J < 0$, $L^2 = \sqrt{-6\alpha J}$:

$$r_e = \frac{L^2}{2\alpha};$$

(iv) $\alpha > 0$, $J < 0$, $L^2 > \sqrt{-6\alpha J}$:

$$r_{e,1} = \frac{L^2 - \sqrt{L^4 + 6\alpha J}}{2\alpha}, \\ r_{e,2} = \frac{L^2 + \sqrt{L^4 + 6\alpha J}}{2\alpha};$$

(v) $\alpha = 0$, $J < 0$, $L \in \mathbf{R}$:

$$r_e = -\frac{3J}{2L^2};$$

(vi) $\alpha < 0$, $J < 0$, $L \in \mathbf{R}$:

$$r_e = \frac{L^2 - \sqrt{L^4 + 6\alpha J}}{2\alpha}.$$

Proof. It is obvious that the relative equilibria are provided by the real, positive roots of equation (9). Solving this equation for the whole interplay among the field parameters α and J, and the angular momentum L, we find only the above stated situations.□

4. STABILITY OF THE RELATIVE EQUILIBRIA

Taking into account the above hypotheses and results, we are in the position to apply the algorithm described by Zombro and Holmes (1993). This technique allows to formulate conclusions about the nonlinear stability of the relative equilibria via the examination of the positive definiteness of the matrix

$$D^2V\left|_{(r=r_e,z=0)}\right. = \begin{bmatrix} \partial^2V/\partial r^2 & \partial^2V/\partial r\partial z \\ \partial^2V/\partial r\partial z & \partial^2V/\partial z^2 \end{bmatrix}_{(r=r_e,z=0)}. \tag{10}$$

The nonlinear stability is entailed by the conditions

$$\begin{aligned} D_1 &:= \left(\partial^2V/\partial r^2\right)_{(r=r_e,z=0)} > 0; \\ D_2 &:= \det\left(D^2V\left|_{(r=r_e,z=0)}\right.\right) > 0. \end{aligned} \tag{11}$$

Taking into account (7) and (9), these conditions read

$$\begin{aligned} D_1 &= \frac{L^2 r_k + 3J}{r_k^5} > 0; \\ D_2 &= \frac{L^2 r_k - 3J}{r_k^5} > 0, \end{aligned} \tag{12}$$

with r_e precised by Proposition 3.3.

In this context we can state the two theorems below, which constitute the main result of this paper.

Theorem 4.1. *The equilibrium solution corresponding to the case (i) is nonlinearly stable for $L^2 > \sqrt{2\alpha J}$. The equilibrium solutions corresponding to the case (ii) and to the circular orbit of radius $r_{e,2}$ of the case (iv) are nonlinearly stable. We can say nothing about the nonlinear stability of the equilibria for the remaining cases.*

Proof. In the case (i), we get $D_1 > 0$ and $D_2 > 0$ for $L^2 > \sqrt{2\alpha J}$ (nonlinear stability), $D_2 \leq 0$ for $L^2 \leq \sqrt{2\alpha J}$ (inconclusive test). In the case (ii), we have $D_1 > 0$, $D_2 > 0$ (nonlinear stability), as well as for the case (iv) with $r_e = r_{e,2}$. For the case (iii), $D_1 = 0 = D_2$, whereas for the cases (iv) with $r_e = r_{e,1}$, (v), and (vi), we obtain $D_1 < 0$, $D_2 < 0$, hence the nonlinear stability test is not conclusive.□

For the situations in which the above test fails, we shall resort to linearization. In this case the following result can be stated:

Theorem 4.2. *The equilibria corresponding to the cases (i) with $L^2 \leq \sqrt{2\alpha J}$, (iii), (iv) with $r_e = r_{e,1}$, (v), and (vi) are linearly unstable.*

Proof. For these cases, the linearized Hamiltonian at the critical point $(r = r_e, z = 0)$ leads straightforwardly to the characteristic equation

$$\lambda^4 + 2\left(D_r + D_z\right)\lambda^2 + 4D_rD_z = 0, \tag{13}$$

where

$$\begin{aligned} 2D_r &= \left(\partial^2V/\partial r^2\right)_{(r=r_e,z=0)} = D_1, \\ 2D_z &= \left(\partial^2V/\partial z^2\right)_{(r=r_e,z=0)} = D_2/D_1 \ \ (\text{for } D_1 \neq 0). \end{aligned} \tag{14}$$

It is easy to check that for the considered cases we have at least one eigenvalue with nonnegative real part; consequently these equilibria are unstable.□

Remark 4.3. It is obvious that the relative rest equilibrium corresponding to the force-free field is linearly unstable, too.

5. CONCLUSIONS

Summarizing, the generalized J_2 problem admits eight families of equilibrium solutions, according to the interplay among the field parameters α and J, and the angular momentum L. For $\alpha > 0$, $J > 0$, the orbits are nonlinearly stable if $L^2 > \sqrt{2\alpha J}$, and unstable for $L^2 \leq \sqrt{2\alpha J}$; the bifurcation values are obtained for $L^2 = \sqrt{2\alpha J}$. For $\alpha > 0$, $J = 0$, the orbits are nonlinearly stable, whatever L is. For $\alpha > 0$, $J < 0$, we have a family of unstable equilibrium orbits for $L^2 = \sqrt{-6\alpha J}$, and two families of equilibrium orbits for $L^2 > \sqrt{-6\alpha J}$: the inner one, at $r = r_{e,1}$ is unstable, whereas the outer one, at $r = r_{e,2}$, is nonlinearly stable. For $\alpha = 0$, $J < 0$, and $\alpha < 0$, $J < 0$, the relative equilibrium orbits are unstable. It is the same for the relative rest equilibrium corresponding to the degenerate case $\alpha = 0$, $J = 0$.

An important result concerning the nonlinearly stable orbits can be stated as

Theorem 5.1. *The circular orbits corresponding to the cases (i) (for $L^2 > \sqrt{2\alpha J}$), (ii), and (iv) (for $r_e = r_{e,2}$) remain stable under the influence of all other zonal harmonics of the potential.*

Proof. The amended potential has in this case the expression

$$U_L = V - \sum_{n\geq 3} \frac{J_n}{\left(\sqrt{r^2+z^2}\right)^{n+1}} P_n\left(\frac{z}{\sqrt{r^2+z^2}}\right). \tag{15}$$

with $z = 0$. One sees that the perturbation preserves the initial symmetry; the value of the momentum p_θ (conjugate to the cyclic coordonate θ) is conserved. But Zombro and Holmes (1993) showed that the nonlinearly stable orbits of the problem remain stable under such perturbations. Accordingly, the circular orbits corresponding to the cases (i) (for $L^2 > \sqrt{2\alpha J}$), (ii), and (iv) (for $r_e = r_{e,2}$) keep their stability.□

Remark 5.2. There are perturbations of different nature which do not affect the nonlinear stability of the given stable equilibrium orbits. The diffusely re-emitted radiation pressure constitutes such an example.

Remark 5.3. The "equatorial" relative equilibrium solutions are not necessarily circular trajectories. One sees that for the cases (i) (with $L^2 < \sqrt{2\alpha J}$) and (vi) they also can reduce to a rest ($L = 0$) with respect to the central body.

Remark 5.4. Taking into account the equivalence between the "equatorial" generalized J_2 problem and the Schwarzschild problem, the relative equilibrium orbits are the same for the two models. The latter model was investigated by Stoica and Mioc (1997), who obtained the global flow and pointed out the relative equilibria, with the same stability features as in the generalized J_2 problem. However, the important results stated by Theorem 5.1 and Remark 5.2 cannot be recovered in the quoted paper.

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СТАБИЛНОСТ РЕЛАТИВНИХ РАВНОТЕЖА У ГЕНЕРАЛИСАНОМ $\mathbf{J}_2$ ПРОБЛЕМУ

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За једну велику класу конкретних астрономских ситуација моделирано је кретање небеских тела у динамичким системима за које важи потенцијална функција $\alpha/r+\varepsilon U$ (r = растојање између честица, α = реална константа, ε = реални мали параметар, U = поремећајни потенцијал). У овом се раду истражује нелинеарна стабилност релативно уравнотежених орбита које одговарају таквом потенцијалу користећи једну мање уобичајену методу а која спаја технику блок- дијагонализације и поступак свођења. Тест истиче извесне нелинеарно стабилне орбите док је неодређен за остале равнотеже. Нелинеарно стабилне орбите остају стабилне при сваком поремећају који задржава конзервисани моменат.